SYMMETRIC NETS

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هذا البحث يهدف إلى دراسة الشبكات المتماثلة، التي تم دراستها لزمن طويل في أشكال مختلفة. وتم عرضها توافقياً وهندسياً وقد استخدم المؤلفون تسميات مختلفة لدراسة الشبكات المتماثلة مثل شبكية التصميم التماثلي المستعرض، نظام هادمارد، والتصميم التماثلي الأفني، والتصميم المستعرض القابل للفك.

في هذه البحث نحن درسنا الشبكات المتماثلة ($\mu, m$) وعلاقتها بالبني التصميمية الأخرى مثل المصروفات المعمول، وتصميم الزمرة القابلة للقسمة، ومصروفات هادمارد، مصروفات هادمارد العمومية، والتصميم المستعرض، كما ناقشنا نوع هام من أنواع الشبكات المتماثلة الذي يسمى بالشبكات المتماثلة الكلاسيكية وحصولنا على التمثيل الجبري للشبكات المتماثلة الكلاسيكية ($1,q$) ثم للشبكات المتماثلة الكلاسيكية ($q,q$) وأخيراً عمنا هذا التمثيل الجبري للشبكات المتماثلة الكلاسيكية ($q^{n-2},q$) حيث $q$ هو قوة لعدد أولي و $2 \geq n$ عدد صحيح.
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ABSTRACT

This research aimed to study the symmetric nets, which have been studied for a long time in various guises and under geometrical or combinatorial presentation.

The authors using different names for examples, a hypernet, symmetric transversal design, Hadamard system, symmetric affine design, and resolvable transversal design. In this research, we studied the \((\mu,m)\) nets and we clarified its relations with other incidence structures like orthogonal arrays, Group Divisible Designs, Hadamard matrices, Generalized Hadamard Matrices, and Transversal Designs. Also we discussed very important type of symmetric nets, which is called classical symmetric net. Also we gave the algebraic representation of the classical \((1,q)\) nets. Then we got the algebraic representation of the classical \((q,q)\) nets, and finally we generalized this to get the algebraic representation of the classical symmetric \((q^{n-2},q)\) nets, where \(q\) is a prime power and \(n \geq 2\) is integer.
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CHAPTER 1

INTRODUCTION

In this chapter, we will study the basic definitions and some important useful results.

1.1 Incidence Structures and t-Designs

Definition 1.1.1. A triple \((V, B, I)\) is called incidence structure, where \(V\) and \(B\) are two non empty disjoint sets and \(I \subseteq V \times B\). The elements of \(V\) are called points, the elements of \(B\) are called blocks, and the elements of \(I\) are called flags.

We say that point \(p\) and a block \(B\) are incident if \((p, B) \in I\). The set of blocks incident with \(p\) denoted by \((p) = \{B \in B : pIB\}\) and \(|(p)|\) is called replication number of \(p\) in general \((Q) := \{B \in B : pIB \forall p \in Q\}\) also if \(C \subseteq B\) then \((c) = \{p \in V : pIB \forall B \in C\}\) and \(|C|\) called the size of \(C\).

Sometimes \(|C|\) called the degree of \(C\) and \(|p|\) called the degree of the point \(p\).

Remarks 1.1.2.

- The incidence structure is finite if \(V\) and \(B\) are finite.

- The incidence structure \((V, B, I)\) is called simple if \((B) \neq (C)\) whenever \(B\) and \(C\) are distinct blocks "in this thesis by incidence structure we mean simple structure".

Definition 1.1.3. An incidence matrix \(M\) of \(\mathcal{D} = (V, B, I)\) is \((0, 1)\)-matrix whose rows are indexed by the points of \(\mathcal{D}\), columns are indexed by the blocks of \(\mathcal{D}\), and the \((p, B)\)-entry is equal to 1 if and only if \((p, B) \in I\).
Remarks 1.1.4.

The incidence matrix $M$ of incidence structure $D$ is mapping of $V \times B$ to $\{0,1\}$. Every matrix with entries from $\{0,1\}$ determines an incidence structure.

Example 1.1.5. Let $V = \{1,2,3,4\}$ be the point set, and block set $B = \{\{1,2\},\{3,4\},\{1,3\},\{2,4\},\{1,4\},\{2,3\}\}$ as incidence relation $I$ the membership relation " $\in$ ". Then $D = (V,B,I)$ is incidence structure and we can find the incidence matrix of $D$ as follows:

$$
M = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
\end{bmatrix}
$$

Definition 1.1.6. The complement structure of an incidence structure $D = (V,B,I)$ is the incidence structure $\overline{D} = (V,B,J)$ with $J = V \times B \setminus I$.

Definition 1.1.7. Two incidence structures $D_1 = (V_1,B_1,I_1)$ and $D_2 = (V_2,B_2,I_2)$ are said to be isomorphic if there exist a bijection $\sigma$ of $V_1 \cup B_1$ onto $V_2 \cup B_2$ which maps $V_1$ on to $V_2$ and $B_1$ on to $B_2$ and preserves incidences and non-incidences (that is $pI_1 B$ if and only if $\sigma(p)I_2\sigma(B)$, where $p \in V_1$, and $B \in B_1$).

An isomorphism of an incidence structure onto itself is called automorphism.

Definition 1.1.8. The dual of $D = (V,B,I)$ is $D^* = (B,V,I^*)$ where $I^* = \{(\beta,p) : (p,\beta) \in I\}$.

An incidence structure $D$ is called self-dual if it is isomorphic to its dual $D^*$. 

Proposition 1.1.9. Let $D$ be an incidence structure with $v$ points and $b$ blocks such that all points have same replication number $r$ and all blocks have same size $k$.

Then $vr = bk$.

Proof: Here we must count all flags as follows:

The $i$-th point is on exactly $r$ block so $vr$ give as the number of flags and in the same way the $j$-th block contains exactly $k$ points hence $bk$ is exactly the number of flags as $vr$.

Hence $vr = bk$.

Definition 1.1.10. Let $D = (V, B, I)$ be an incidence structure. Let $X_0$ be an empty subset of $V$ and $B_0$ a non-empty subset of $B$. The incidence structure $D(X_0, B_0) = (X_0, B_0, I \cap (X_0 \times B_0))$ is said to be substructure of $D$, and if $B = B_0$ then it is denoted by $D(X_0)$ instead of $D(X_0, B)$.

If $N$ is the incidence matrix of $D$. Then the submatrix of $N$ formed by the rows with incidence from $X_0$ and columns with incidence from $B_0$ is an incidence matrix of $D(X_0, B)$.

Definition 1.1.11. Let $D = (V, B, I)$ be incidence structure and let $Y$ be a proper subset of $V$. Let $B^Y = \{B \in B : Y \not\subseteq B\}$ and $B_Y = \{B \in B : B \not\subseteq Y\}$, if $B^Y \neq \Phi$, then the substructure $D^Y = D(V \setminus Y, B^Y)$ is called residual substructure of $D$. If $B_Y \neq \Phi$, then the substructure $D_Y = (Y, B_Y)$ is called a derived substructure of $D$. If $Y$ is the set of all points incident with a block $B$ then we write $D^B$ and $D_B$ instead of $D^Y$ and $D_Y$ respectively, then we call these substructure block-residual.
and block-derived. If \( Y \) is a singleton \( y \) then we put \( D^Y = D^\{y\} \) and
\( D_Y = D_{V \setminus \{y\}} \)
and call these substructures point-residual and point-derived respectively.

**Theorem 1.1.12.** ([BJL: 99]) A simple incidence structure on \( v \) points with
\( b \) blocks, constant block degrees \( k \) and constant point degrees \( r \) exist if and only if
\( vr = bk \) and \( b \leq \binom{v}{k} \).

**Remark 1.1.13.** If an incidence structure \((V, B, I)\) such that \( B \) is a set of
subset of \( V \) and \((x, B) \in I\) if and only if \( x \in B \), then it will denoted as \((V, B)\).

**Definition 1.1.14.** A \( t - (v, k, \lambda) \) design where \( v > k \geq 1, t \geq 0 \) and \( \lambda \geq 0 \) is a
finite incidence structure \((V, D)\), such that:

1. \(|V| = v\)
2. \(D\) is a family of \( k\)-subset of \( V \) (that is \( \forall B \in D \quad |B| = k \)).
3. Every \( t\)-subset of \( V \) is contained in \( \lambda \) members of \( D \), and the number of
   blocks is denoted by \( b \).

**Definitions 1.1.15.**

1. A \( t - (v, k, \lambda) \) design with \( \lambda = 1 \) is called a Steiner design.
2. A \( t - (v, k, \lambda) \) design with \( t = 1 \) is called tactical conﬁguration.
3. A \( t - (v, k, \lambda) \) design with \( t = 2 \) is called balanced incomplete block design in
   short (BIBD).
4. A \( t - (v, k, \lambda) \) design is called trivial if every \( k\)-subset of \( V \) is a block i.e., if
   \( b = \binom{v}{k} \).
**Lemma 1.1.16.** Let $V$ be a finite set and $D$ is the set of all $k$—subsets of $V$, for some positive integer $k$. Then $(V, D)$ is a $t - (v, k, \binom{v-t}{k-t})$ design for every $1 \leq t \leq k - 1$.

**Proof.**

Let $A$ be any $t$—subset of $V$, then $A$ can be completed to $k$—subset of $V$ by $\binom{v-t}{k-t}$ ways, therefore every such $A$ is in $\binom{v-t}{k-t}$ blocks in $D$. Since this number is independent of $A$, therefore $(V, D)$ is a $t - (v, k, \binom{v-t}{k-t})$ design.

**Corollary 1.1.17.** If $t - (v, k, \lambda)$ is not trivial design then:

1. $|B| < \binom{v}{k}$.
2. $\lambda < \binom{v-t}{k-t}$.

**Example 1.1.18.** Let $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $B = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}, \{1, 5, 9\}, \{2, 6, 7\}, \{3, 4, 8\}, \{1, 6, 8\}, \{2, 4, 9\}, \{3, 5, 7\}\}$.

We can check the conditions of $t$—design so $(V, B)$ is finite incidence structure and each point of $V$ is in 4 blocks of $B$ i.e., $(V, B)$ is $1 - (9, 3, 4)$ design or $3 - (9, 3, 1)$.

**Example 1.1.19.** Let $V = \mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ and $B = \{\{0, 2, 6\}, \{0, 4, 5\}, \{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{1, 5, 6\}, \{3, 4, 6\}\}$. It can be checked that each two points of $V$ exist in one block or every point of $V$ exist in three blocks, i.e., $(V, B)$ is $2 - (7, 3, 1)$ or $1 - (7, 3, 3)$. This design is very important and known as Fano plane or projective plane of order two.

**Example 1.1.20.** Let $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{\{1, 3, 7, 8\}, \{1, 2, 4, 8\}, \{2, 3, 5, 8\}\{3, 4, 6, 8\}, \{4, 5, 7, 8\}, \{1, 5, 6, 8\},$
\{2, 6, 7, 8\}, \{1, 2, 3, 6\}, \{1, 2, 5, 7\}, \{1, 3, 4, 5\}, \{1, 4, 6, 7\}, \{2, 3, 4, 7\}, \{2, 4, 5, 6\},
\{3, 5, 6, 7\} is 2 - (8, 4, 3) or 1 - (8, 4, 7).

**Theorem 1.1.21.** Let \( D \) be \( t-(v,k,\lambda) \) design, then \( D \) also \( s-(v,k,\lambda_s) \)
where \( 0 \leq s \leq t \).

And \( \lambda_s = \lambda \frac{(v-s)}{(t-s)} \frac{(k-s)}{(t-s)} \).

**Proof :**

Let \( A \) be \( s \)-Subset of \( V \) and let \( X \) be \( (t-s) \)-Subset of \( V \) where \( A \) and \( X \) are disjoint sets.

Now we can count the pair \((X, B)\) where \( B \) is a block such that \( X \cup A \subseteq (B) \) in two ways.

First, since there are \( \binom{v-s}{t-s} \) ways to choosing \( X \) and for each choice of \( X \) there are \( \lambda \) choices of \( B \). Then there exist \( \lambda \binom{v-s}{t-s} \) pairs \( (*) \).

Second way to count the pairs \((X, B)\) for given \( B \) with \( A \subseteq (B) \), there are \( \binom{k-s}{t-s} \) ways to choosing \( X \), hence we obtain \( \lambda_s \binom{k-s}{t-s} \) pairs \( (**) \).

From \( (*) \) and \( (**) \) we get \( \lambda \binom{v-s}{t-s} = \lambda_s \binom{k-s}{t-s} \).
Corollary 1.1.22.

1. From the definition of \( t \)-design we get \( \lambda_1 \) is the replication number of any point in \( V \) denoted by \( r \), and \( \lambda_0 \) is the number of blocks \( b \) so \( \lambda \left( \begin{array}{c} v-1 \\ t-1 \end{array} \right) = r \left( \begin{array}{c} k-1 \\ t-1 \end{array} \right) \).

2. If \( t = 2 \), then \( \lambda(v-1) = r(k-1) \).

3. Any \( t-(v,k,\lambda) \) can be described as \( 1-(v,k,r) \) where, \( t > 0 \).

The proof can be get easily from Theorem 1.1.21.

Proposition 1.1.23. If \( D \) is \( t-(v,k,\lambda) \) design then \( bk = vr \).

Proof: Easily follows from proposition 1.1.9.

Theorem 1.1.24. (Fisher’s inequality)(see [BJL: 99])

For any \( t-(v,k,\lambda) \) design, the number of points does not exceed the number of blocks, i.e., \( v \leq b \).

As we defined the derived and the residual incidence structure we can define this concept in \( t-(v,k,\lambda) \) design.

Definition 1.1.25. If \( D = (V,B) \) is \( t-(v,k,\lambda) \) design and \( p \in V \), then the \( D_p \) is the incidence structure whose points set is \( V - \{p\} \), and blocks set is the blocks of \( D \) which incident with \( p \), is called a derived design, and \( D_p \) is a \( (t-1)-(v-1,k-1,\lambda) \) design.

Definition 1.1.26. If \( (V,B) \) is a \( t-(v,k,\lambda) \), then \( (V,\overline{B}) \) where \( \overline{B} = \{v-c\} : c \in B \} \) is called the complement design of \( D \).
**Result 1.1.27.** Using the incidence matrix we can express any design which is easy method to know the properties of the design.

By using the incidence matrix $M$ of the design $D$ we can get the incidence matrix of $\tilde{D}$ by interchange any 0 to 1, and any 1 to 0 in the matrix $M$.

Also to get the incidence matrix of $(V, B)$, we see to the incidence matrix of $(V, B)$, and delete any column has a 0 in the $i$th row, then delete the $i$th row we get the incidence matrix of $(V, B)$.

**Theorem 1.1.28.** If $D$ is $2-(v, k, \lambda)$ design then its complement is $2-(v, v-k, b-2r+\lambda)$, where $b-2r+\lambda > 0$ with $\overline{b} = b$, $\overline{r} = r$.

**Example 1.1.29.** Let $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{\{1, 3, 7, 8\}, \{1, 2, 4, 8\}, \{2, 3, 5, 8\}, \{3, 4, 6, 8\}, \{4, 5, 7, 8\}, \{1, 5, 6, 8\}, \{2, 6, 7, 8\}, \{1, 2, 3, 6\}, \{1, 2, 5, 7\}, \{1, 3, 4, 5\}, \{1, 4, 6, 7\}, \{2, 3, 4, 7\}, \{2, 4, 5, 6\}, \{3, 5, 6, 7\}\}$ is $2-(8, 4, 3)$ or $1-(8, 4, 7)$.

We can find the complement $(V, \overline{B})$ design of the $2-(8, 4, 3)$ design as follows:

The point set is $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and

$\overline{B} = \{\{2, 4, 5, 6\}, \{3, 5, 6, 7\}, \{1, 4, 6, 7\}, \{1, 2, 5, 7\}, \{1, 2, 3, 6\}, \{2, 3, 4, 7\}, \{1, 3, 4, 5\}, \{4, 5, 7, 8\}, \{3, 4, 6, 8\}, \{2, 6, 7, 8\}, \{2, 3, 5, 8\}, \{1, 5, 6, 8\}, \{1, 3, 7, 8\}, \{1, 2, 4, 8\}\}$ which is $2-(8, 5, 14-2*7+3)$ i.e., $2-(8, 5, 3)$ by using Theorem 1.1.28, also we can check this by definition of $t-$ design.

And we can solve the above example easily by incidence matrices as follows:

Since
We get the incidence matrix of the complement design, by change 0 to 1 and 1 to 0 in the matrix $M$ as follows:

$$
M = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Example 1.1.30. Let $(V, B)$ be the $2-(7, 3, 1)$ design where $V = \{0, 1, 2, 3, 4, 5, 6\}$.

And let $B = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{0, 4, 5\}, \{1, 5, 6\}, \{0, 2, 6\}\}$

Let $p = 1$. then we can get the derived design $(V, B)_1$