MARANGONI CONVECTIONS IN A HORIZONTAL POROUS LAYER
SUPERPOSED BY A FLUID LAYER IN THE PRESENCE OF
UNIFORM VERTICAL MAGNETIC FIELD AND SOLUTE

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ABSTRACT

A linear stability analysis applied to a system consisting of a horizontal fluid layer overlying a porous layer in the presence of uniform vertical magnetic field and solute on both layers. Flow in porous medium is assumed to be governed by Darcy's law, where as the flow in the fluid layer is governed by Navier-Stokes equation. The Beavers-Joseph condition is applied at the interface between the two layers. Numerical solutions are obtained for stationary convection case using the method of expansion of Chebyshev polynomials. It is found that the spectral method has a strong ability to solve the multi-layered problem, the depth ratio has a strong effect and that the magnetic field has effect in this model.

Keywords: Navier-Stokes equation, Darcy's law, Chebyshev polynomials, Magnetic equations, Maragoni Convections, Magnetic field and solute, Non-dimensionalization.

1 INTRODUCTION

Thermal instability theory has attracted considerable interest and has been recognized as a problem of fundamental importance in many fields of fluid dynamics. The earliest experiments to demonstrate the onset of thermal instability in fluids, are those of Benard (Benard 1900; Benard 1901). Benard worked with very thin layers of an incompressible viscous fluid, standing on leveled metallic plate maintained at a constant temperature. The upper surface was usually free and being in contact with the air was at a lower temperature. In his experiments Benard deduced that a certain critical adverse temperature gradient must be exceeded before instability can set in. Rayleigh (Rayleigh 1916) provided a theoretical basis for Benard's experimental results. He showed that the numerical value of the non-dimensional parameter, Jeffrey's (Jeffreys 1930) showed that the onset of thermal instability criterion derived for incompressible fluids could be used for compressible fluids under some certain conditions on the adverse temperature gradient. Benard convection in the context of magneto hydrodynamic fluid has been examined by Thompson (Thompson 1951) and Chandrasekhar (Chandrasekhar 1952) with a linear constitutive relationship between the magnetic field and the magnetic induction. Nield (Nield 1968) considered the onset of salt-finger convection in a porous layer. Taunton et al., (Taunton and

Lightfoot (1972) considered the thermohaline instability and salt-fingers in a porous medium and solved the boundary value problem. Sun (Sun 1973) was the first to consider such a problem, and he used a shooting method to solve the linear stability equations. The results show that the critical Raleigh number in the porous layer decreases continuously as the thickness of the fluid layer is increased. Nield (Nield 1977) formulated the problem with surface-tension effects at a deformable upper surface include and obtained asymptotic solution for small wave numbers for a constant heat-flux boundary condition. Sun (Sun 1973) and Nield (Nield 1977) used Darcy’s law in formulating the equations for porous layer. In Darcy’s low of motion in porous mediums, the Darcy resistance term took the place of the Navier-Stokes viscose term while in modified Darcy’s law (Brinkman model) that suggested by Brinkman (Brinkman 1947a; Brinkman 1947b) the Navier-Stokes viscous term still exists. Chandrasekhar (Chandrasekhar 1981) considered typical problems in hydrodynamic and hydro magnetic stability in his treatise. Somerton and Catton (Somerton and Catton 1982) have different formulation of the problem from Sun (Sun 1973) and Nield (Nield 1977) by inclusion Brinkman term in Darcy equation for the porous layer and solved the problem using Galerkin method. Hilles (Hills et al. 1983) and Maples & Poirier, (Maples and Poirier 1984) discussed the stability of the boundary value problem of a thermo dynamical consistent model, the directionnal solidification of molten alloys as a layer of porous material of variable permeability which is separated from its melt by a mush zone of dendrites. Glicksman (Glicksman et al. 1986) described the interaction between the solidifying alloy and its melt by a doubly diffusive model. Chen & Chen (Chen and Chen 1988) considered the multi-layer problem when the above layer is heated and salted from above, and solution of problem is obtained using a shooting method. Abdullah (Abdullah 1991) discussed the Benard convection in a non-linear magnetic field under the influence of vertical and non-vertical magnetic field for different boundary conditions. Lindsay & Ogden (Lindsay and Ogden 1992) worked in the implementation of spectral methods resistant to the generation of spurious Eigenvalues. Lamb (Lamb 1994) used expansion of Chebyshev polynomials investigate an Eigenvalue problem arising from a model discussing the instability of the earth’s core. Bukhari (Bukhari 1996) studied the effects of surface-tension in a layer of conducting fluid with imposed magnetic field and obtained results when deformable and non-deformable free surface and he solved by using Chebyshev spectral method, and he obtained some different results from that of Chen & Chen (Chen and Chen 1988). Straughan (Straughan 2001) studied the thermal convection in fluid layer overlying a porous layer and he considered when the problem of lower layer heated from below and surface tension driven on the free top boundary of upper layer. Also, in (Straughan 2002) he dealt with the same problem, considering the ratio depth of the relative layer also, investigated the effect of the variation of relevant fluid and porous material properties. Bukhari (Bukhari 2003a) applied the linear stability analysis in the system consisting of a horizontal fluid layer overlying a layer of a porous medium affected a vertical magnetic field on both layers. The same author (Bukhari 2003b) applied the spectral Chebyshev polynomial method to obtain numerically the solution of a multi-layer system consisting of the finger convection onset in a fluid layer overlying a porous layer, and he studied the boundary value problem when the thermal Rayleigh number and the critical salt Rayleigh number for porous layer are increasing due to heating and salting the lower of porous layer.

In this paper we studies convective instabilities in a horizontal multi-layer problem in the presence of uniform vertical magnetic field and solute. i.e., we shall consider the onset of thermal convection in a horizontal porous layer superposed by a fluid layer affected by a vertical magnetic field and solute. The flow in the porous layer is assumed to be governed by Dray’s law. The linear stability equations will be solved using expansion of Chebyshev polynomials. This

method is better suited to the solution of hydrodynamic stability problems than expansions in other sets of orthogonal polynomials. Lanczos (Lanczos 1938) is the first presented of this method. This method have used and developed to solve ordinary differential equations by Fox (Fox 1962), Fox and Parker (Fox and Parker 1968), Orszag (Orszag 1971a; Orszag 1971b). Chaves & Ortiz (Chaves and Ortiz 1968) applied it to a second order Eigenvalue problem with polynomial coefficients. Hassaniien & El-Hawary (Hassaniien and El-Hawary 1990) studied Chebyshev solution of laminar boundary layer flow. Bukhari (Bukhari 1996) has used this method to solve multilayers region. Hassaniien et. al., (Hassaniien et al. 1996) studied axisymmetric stagnation flow on a cylinder using series expansions of Chebyshev polynomials.

2 MATHEMATICAL FORMULATION

Consider the problem of two horizontal layers \( L_1 \) and \( L_2 \) such that the bottom of the layer \( L_1 \) touches the top of the layer \( L_2 \). A right handed system of Cartesian Coordinates \((x_i, i = 1, 2, 3)\) is chosen so that the interface is the plan \( x_3 = 0 \), the top boundary of \( L_1 \) is \( x_3 = \delta_f + F(t, \alpha) \), and the lower boundary of \( L_2 \) is \( x_3 = -d_m \) (see Fig. (1)). Suppose that the upper layer \( L_1 \) is filled with an incompressible thermally and electrically conducting viscous fluid consisting of melted solute which flow in it and is governed by Navier-Stokes equations, where as the lower layer is occupied by a porous medium permeated by the fluid which flow in it and is governed by Darcy’s law and is subjected to a uniform vertical magnetic field. Gravity \( g \) acts in the negative direction of \( x_3 \), the heated and solutal from below.

![Figure 1: Schematic diagram of the problem.](image)

Convection is driven by temperature dependence of the fluid density and soluting, and damped by viscosity. The Oberbeck-Boussinesq approximation is used as density of the fluid is constant everywhere except in the body force term where the density is linearly proportional to

temperature and solute concentration.

\[ \rho_f = \rho_0 [1 - \alpha(T - T_0) + \alpha'(S - S_0)], \]  

where \( T \) denotes the Kelvin temperature of the fluid, \( S \) is the solute concentration, \( \rho_0 \) is density of the fluid at \( T_0 \) and \( S_0 \), \( \alpha \) (constant) is the thermal coefficient of volume expansion of the fluid and \( \alpha' \) (constant) is the soluting coefficient of volume expansion of the fluid.

Furthermore, incompressibility of the fluid and the non-existence of magnetic monopoles require that \( \mathbf{V} \) and \( \mathbf{B} \) are both solenoidal vectors. Hence

\[ \text{div} \mathbf{V} = V_{t,i} = 0, \quad \text{div} \mathbf{B} = B_{t,i} = 0. \]  

Suppose also that the magnetization in the fluid is directly proportional to the applied field that the fluid behaves like an Ohmic conductor so that the magnetic field \( \mathbf{H} \), magnetic induction \( \mathbf{B} \), current density \( \mathbf{J} \) and electric field \( \mathbf{E} \) are connected by the constitutive relations

\[ \mathbf{B} = \mu_m \mathbf{H}, \]  
\[ \mathbf{J} = \sigma (\mathbf{E} + \nabla \times \mathbf{B}), \]  

and the Maxwell equations

\[ \text{curl} \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t}, \]  

\[ \mathbf{J} = \frac{1}{4\pi} \text{curl} \mathbf{H}, \]  

where \( \mu_m \) (constant) is the magnetic permeability, \( \sigma \) is the electrical conductivity and the displacement current has been neglected in the second of these Maxwell equations as is customary in situations when free charge is instantaneously dispersed. On taking the \( \text{curl} \) of equation (4) and replacing the electric field by the Maxwell relation (5), the magnetic field \( \mathbf{H} \) is now readily seen to satisfy the partial differential equation

\[ \eta \text{curl} \text{curl} \mathbf{H} = -\frac{\partial \mathbf{H}}{\partial t} + \text{curl}(\mathbf{V} \times \mathbf{H}), \]  

where \( \eta = \frac{1}{4\pi \sigma} \) is the electrical resistivity. In addition, the constant nature of \( \mu_m \) makes the magnetic field \( \mathbf{H} \) a solenoidal vector. Equation (7) is now reworked using standard vector identities to yield in sequence

\[ \frac{\partial \mathbf{H}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{H} = (\mathbf{H} \cdot \nabla) \mathbf{V} + \eta \nabla^2 \mathbf{H}, \]  

equation (8) describes the temporal evolution of magnetic field. Moreover, the relations (3), (4), (5) and (6) can be used to recast the Lorentz force \( \mathbf{J} \times \mathbf{B} \) into

\[ \mathbf{J} \times \mathbf{B} = \frac{\mu_m}{4\pi} ((\mathbf{H} \cdot \nabla) \mathbf{H} - \frac{1}{2} \nabla \mathbf{H}^2), \]  

leading to the notion that the Lorentz force is derived from a magnetic stress tensor \( \sigma_{j,k}^{(m)} \). In view of the fact that

\[ (\mathbf{J} \times \mathbf{B})_i = \sigma_{j,i}^{(m)} = \frac{\mu_m}{4\pi} (H_j H_i - \frac{1}{2} H^2 \delta_{ij}) \]  

so the governing equations of the fluid layer are

\[ \rho_0 \left( \frac{\partial \mathbf{V}_f}{\partial t} + \mathbf{V}_f \cdot \nabla \mathbf{V}_f \right) = -\nabla P_f + \mu \nabla^2 \mathbf{V}_f + \rho_f \mathbf{g} + \frac{\mu_{mf}}{4\pi} (\mathbf{H}_f \cdot \nabla \mathbf{H}_f), \quad (10) \]

\[ (\rho \, c_p)_f \frac{\partial T_f}{\partial t} + \mathbf{V}_f \cdot \nabla T_f = k_f \nabla^2 T_f, \quad (11) \]

\[ \frac{\partial S_f}{\partial t} + \mathbf{V}_f \cdot \nabla S_f = M_f \nabla^2 S_f, \quad (12) \]

\[ \frac{\partial H_f}{\partial t} = (\mathbf{H}_f \cdot \nabla) \mathbf{V}_f - (\mathbf{V}_f \cdot \nabla) \mathbf{H}_f + \eta_f \nabla^2 \mathbf{H}_f, \quad (13) \]

and the governing equations of the porous medium layer are

\[ \rho_0 \frac{\partial \mathbf{V}_m}{\partial \varphi} = -\nabla P_m - \frac{\mu}{K} \mathbf{V}_m + \rho_f \mathbf{g} + \frac{\mu_{mm}}{4\pi} (\mathbf{H}_m \cdot \nabla \mathbf{H}_m), \quad (14) \]

\[ (\rho \, c_m) \frac{\partial T_m}{\partial \varphi} + (\rho \, c_p)_f \frac{\partial T_f}{\partial \varphi} \mathbf{V}_m \cdot \nabla T_m = k_m \nabla^2 T_m, \quad (15) \]

\[ \varphi \frac{\partial S_m}{\partial t} + \mathbf{V}_m \cdot \nabla S_m = M_m \nabla^2 S_m, \quad (16) \]

\[ \frac{\partial H_m}{\partial t} = (\mathbf{H}_m \cdot \nabla) \mathbf{V}_m - (\mathbf{V}_m \cdot \nabla) \mathbf{H}_m + \eta_m \nabla^2 \mathbf{H}_m, \quad (17) \]

where \( P_f, P_m \) are the pressure of the fluid and the porous medium layers respectively; \( \eta_f = \frac{1}{\sigma_f \mu_{mf}}, \eta_m = \frac{1}{\sigma_m \mu_{mm}} \) are the electrical resistivity of the fluid and the porous layer respectively; \( \mathbf{V}_f, \mathbf{V}_m \) are the solenoidal and seepage velocity respectively; \( T_f, T_m \) are the Kelvin temperature of the fluid and porous medium layer respectively; \( S_f, S_m \) are solute concentration of the fluid and porous medium layer respectively; \( M_f, M_m \) are the mass diffusivity of the fluid and porous medium layer respectively; \( \mu_{mf}, \mu_{mm} \) are magnetic permeability of fluid and porous layer respectively; \( k_f, k_m \) are the thermal conductivity of fluid and porous layer respectively; \( \mu \) is the dynamic viscosity; \( K \) is the permeability; \( \varphi \) is the porosity and \((\rho \, c_p)_f, (\rho \, c_m)\) are the heat and overall heat capacity per unit volume of the fluid and porous medium layers at constant pressure. In fact

\[ (\rho \, c)_m = \varphi (\rho \, c_p)_f + (1 - \varphi) (\rho \, c_p)_m, \]

where \((\rho \, c_p)_m\) is the heat capacity per unit volume of the porous substrate.

3 BOUNDARY CONDITIONS

For the upper boundary we shall suppose \( x_3 = d + F(t, x_3) \) where \( t \) is the time with unit normal \( n = n_i e_i \) directed from the viscous fluid into the passive inviscid fluid. So, the boundary conditions come from following sources.

General radiation conditions the heat and mass transfer:

\[ T_i n_i + L(T - T_\infty) = 0, \quad (18) \]

\[ S_i n_i + C(S - S_\infty) = 0, \quad (19) \]

where \( L, C \) constants. Material surface particles of the fluid remain on the surface at \( x_3 = d + F(t, x_\alpha) \) and so

\[
\frac{dx_3}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_\alpha} \frac{\partial x_\alpha}{\partial t},
\]

this leads to the condition

\[
V_3 - \frac{\partial F}{\partial x_\alpha} V_\alpha = \frac{\partial F}{\partial t}.
\tag{20}
\]

Magnetic condition since the region \( x_3 > d_f \) is electrically insulating, then \( B_i = \mu_{mf} H_i \) is continuous across \( x_3 = d_f \) and the magnetic field in \( x_3 > d_f \) is irrotational, that is, derived from a potential function.

Stress conditions which can be decomposed further into the tangential and normal components respectively,

\[
\tau(T, S) b^o_a = -P_\infty - (P + \frac{\mu_{mf}}{\varepsilon} H_i^2) + 2 \rho_0 n_i n_j + \frac{\mu_{mf}}{\varepsilon} (H_i n_j)^2,
\tag{21}
\]

\[
\tau_{i,\alpha} = \rho_0 n_j (V_{i,j} + V_{j,i}) + \frac{\mu_{mf}}{4\pi} (H_i n_j)(H_j x_{i,\alpha}),
\tag{22}
\]

where \( \nu \) is kinematics viscosity. At the interface \( x_3 = 0 \) we have

\[
T_f = T_m, \ S_f = S_m, \ k_f \frac{\partial T_f}{\partial x_3} = k_m \frac{\partial T_m}{\partial x_3}, \ M_f \frac{\partial S_f}{\partial x_3} = M_m \frac{\partial S_m}{\partial x_3}, \ \omega_f = \omega_m,
\]

\[-P_f + 2\mu \frac{\partial \omega_f}{\partial x_3} = -P_m, \ H_f = H_m, \ k_f \frac{\partial H_f}{\partial x_3} = k_m \frac{\partial H_m}{\partial x_3},
\tag{23}
\]

\[
\frac{\partial u_f}{\partial x_3} = \alpha_{BJ} \frac{\sqrt{R}}{\nu_f - \nu_m}, \ \frac{\partial u_f}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{R}} (\nu_f - \nu_m),
\]

where \( \omega_f \) and \( \omega_m \) are the normal axial velocity components of the fluid in fluid layer and porous medium layer respectively; \( u_f, \nu_f \) are the limiting tangential component of the fluid velocity as the interface is approached from the fluid layer \( L_1 \), whereas \( u_m, \nu_m \) are the same limiting components of tangential fluid velocity as the interface is approached from the porous layer \( L_2 \); the last two boundary conditions in (23), discontinuities in shear velocity across the interface are inherent, also they calls Beavers & Joseph (Beaver and Joseph 1967). \( \alpha_{BJ} \) is Beavers and Joseph constant.

At the lower boundary \( x_3 = -d_m \) we have

\[
\omega_m(-d_m) = 0, \ T_m(-d_m) = T_i, \ S_m(-d_m) = S_i, \ B_i = \mu_{mm} H_i \text{ be continuous.}
\tag{24}
\]

At equilibrium case

\[
V_f = 0, \ V_m = 0, \ -\nabla P_f + \rho_f g = 0, \ -\nabla P_m + \rho_f g = 0,
\]

\[
\nabla^2 T_f = \nabla^2 T_m = 0, \ \nabla^2 S_f = \nabla^2 S_m = 0, \ \ F(t, x_\alpha),
\tag{25}
\]

\[
\nabla^2 H_f = \nabla^2 H_m = 0, \ H = (0, 0, H) : H \text{ constant.}
\]

The boundary conditions in the exterior are

\[
T_f(d_f) = T_u, \ T_m(-d_m) = T_i, \ S_f(d_f) = S_u, \ S_m(-d_m) = S_i,
\tag{26}
\]

and the interface conditions

\[ T_f(0) = T_m(0), \quad S_f(0) = S_m(0), \quad k_f \frac{\partial T_f(0)}{\partial x_3} = k_m \frac{\partial T_m(0)}{\partial x_3}, \]

\[ M_f \frac{\partial S_f(0)}{\partial x_3} = M_m \frac{\partial S_m(0)}{\partial x_3}, \quad P_f(0) = P_m(0), \quad H_f(0) = H_m(0), \]

\[ k_f \frac{\partial H_f(0)}{\partial x_3} = k_m \frac{\partial H_m(0)}{\partial x_3}, \]

are the equilibrium of temperature field, solute concentration, hydrostatic pressure and magnetic field in the fluid layer and porous medium layer respectively, where

\[ T_f = T_0 - (T_o - T_u) \frac{x_3}{d_f}, \quad S_f = S_0 - (S_o - S_u) \frac{x_3}{d_f}, \quad P_f = P_f(x_3), \]

\[ H_f = (0, 0, H), \quad 0 \leq x_3 \leq d_f; \]

\[ T_m = T_0 - (T_o - T_u) \frac{x_3}{d_m}, \quad S_m = S_0 - (S_o - S_u) \frac{x_3}{d_m}, \quad P_m = P_m(x_3), \]

\[ H_m = (0, 0, H), \quad -d_m \leq x_3 \leq 0; \]

where the interfacial temperature and solute concentration are determined by the continuity of heat flux and solute flux respectively and take the values

\[ T_0 = \frac{k_f d_m T_u + k_m d_f T_1}{k_f d_m + k_m d_f}, \quad S_0 = \frac{M_f d_m S_u + M_m d_f S_i}{M_f d_m + M_m d_f}. \]

4 PERTURBED EQUATIONS

Suppose that the equilibrium solution be perturbed by following linear perturbation quantities

\[ \tilde{V}_f = 0 + \tilde{v}_f, \quad \tilde{P}_f = P_f(x_3) + \tilde{p}_f, \quad \tilde{H}_f = (0, 0, H) + \tilde{h}_f, \]

\[ T_f = T_0 - (T_o - T_u) \frac{x_3}{d_f} + \tilde{\theta}_f, \quad S_f = S_0 - (S_o - S_u) \frac{x_3}{d_f} + \tilde{s}_f, \]

\[ \tilde{V}_m = 0 + \tilde{v}_m, \quad \tilde{P}_m = P_m(x_3) + \tilde{p}_m, \quad \tilde{H}_m = (0, 0, H) + \tilde{h}_m, \]

\[ T_m = T_0 - (T_1 - T_0) \frac{x_3}{d_m} + \tilde{\theta}_m, \quad S_m = S_0 - (S_1 - S_0) \frac{x_3}{d_m} + \tilde{s}_m. \]

Then we may verify that the Linearized version of equations (10), (11), (12) and (13) are

\[ \rho_0 \left[ \frac{\partial \tilde{V}_f}{\partial t} + \tilde{V}_f \cdot \nabla \tilde{V}_f \right] = -\nabla \tilde{P}_f + \mu \nabla^2 \tilde{V}_f + \rho_0 \alpha \tilde{s}_f \tilde{g} \]

\[ \quad - \rho_0 \alpha \tilde{s}_f \tilde{g} + \frac{\mu_{nf} \tilde{h}_f}{4n} \frac{\partial h_f}{\partial x_3} + \tilde{h}_f \cdot \nabla \tilde{h}_f, \]

\[ \rho_c v_f \left[ \frac{\partial \tilde{\theta}_f}{\partial t} + \tilde{V}_f \cdot \left( \nabla \tilde{\theta}_f - \frac{(T_o - T_u)}{d_f} e_3 \right) \tilde{e}_3 \right] = k_f \nabla^2 \tilde{\theta}_f, \]

\[ \frac{\partial \tilde{s}_f}{\partial t} + \tilde{V}_f \cdot \left( \nabla \tilde{s}_f - \frac{(S_o - S_u)}{d_f} e_3 \right) = M_f \nabla^2 \tilde{s}_f, \]

\[ \frac{\partial \theta_f}{\partial t} = H_f \frac{\partial \nu_f}{\partial x_3} + (h_f \cdot \nabla) \nu_f - (\nu_f \cdot \nabla) h_f + \eta_f \nabla^2 h_f, \]  

and equations (14), (15), (16) and (17) are

\[ \frac{\rho_0}{\varphi} \frac{\partial \nu_m}{\partial t} = -\nabla p_m - \frac{\mu_c}{K} \nu_m + \rho_0 \alpha \theta_m e_3 \]
\[ - \rho_0 \alpha \theta_m s_m e_3 + \frac{\mu_{mn}}{4\pi} \left[ H_m \frac{\partial h_m}{\partial x_3} + h_m \cdot \nabla h_m \right], \]  

\[ (\rho c)_m \frac{\partial \theta_m}{\partial t} + (\rho c)_m \nabla \cdot \left( \nu_m \left( \nabla \theta_m - \frac{(T_i - T_0)}{d_m} e_3 \right) \right) = k_m \nabla^2 \theta_m, \]

\[ \varphi \frac{\partial s_m}{\partial t} + \nu_m \cdot \left( \nabla s_m - \frac{(S_i - S_0)}{d_m} e_3 \right) = M_m \nabla^2 s_m, \]

\[ \frac{\partial h_m}{\partial t} = H_m \frac{\partial \nu_m}{\partial x_3} + (h_m \cdot \nabla) \nu_m - (\nu_m \cdot \nabla) h_m + \eta_m \nabla^2 h_m, \]

where \( e_3 \) is the unit vector in the \( x_3 \)-direction. \( \nu_f, \nu_m \) are solenoidal.

The boundary conditions, on the upper boundary \( x_3 = d_f + F(t, x_\alpha) \), the conditions (18), (19) and (20) become

\[ (n_3 - 1)(T_u - T_0) + d_f \nu_i \theta_i + Nu \theta_f + Nu(T_u - T_0) \frac{F}{d_f} = 0, \]

\[ (n_3 - 1)(S_u - S_0) + d_f \nu_i s_i + Nu_s s_f + Nu_s(S_u - S_0) \frac{F}{d_f} = 0, \]

\[ \nu_3 \frac{\partial F}{\partial x_\alpha} = \frac{\partial F}{\partial t}, \]

where \( Nu \) and \( Nu_s \) are Nusselt numbers

\[ Nu = Ld_f, \quad Nu_s = Cd_f. \]

The surface stress conditions (21), (22), the normal component and the tangential surface stress are

\[ \tau(T, S) b_3^G = -p_f + \frac{\mu_{mf}}{8\pi} \left( (h_f n_f)^2 + 2H_f n_3 h_f n_j \right) + \rho_0 \mu \frac{G}{F} \]
\[ - \frac{\mu_{mf}}{8\pi} \left( H_f - n_3^2 \right) + 2\rho_0 \nu n_i j n_i n_j \]
\[ - \rho_0 \mu \frac{G}{2d_f} \left[ \alpha(T_u - T_0) - \alpha(S_u - S_0) \right] F^2, \]

\[ \frac{\partial \tau}{\partial T} \tau_{x_\alpha} + \frac{\partial \tau}{\partial S} x_3 = \rho_0 \nu (n_i j + n_i j) n_i x_i x_\alpha + \frac{\mu_{mf}}{4\pi} (H_f n_3 + h_f n_j) (h_i n_i x_\alpha) \]

In (41) and (42), it is assumed that the derivative of the surface tension with respect to temperature and concentration are evaluated at

\[ T = T_u + (T_u - T_0)d_f^{-1} F + \theta_f, \quad S = S_u + (S_u - S_0)d_f^{-1} F + s_f. \]

whereas the boundary conditions on the interface $x_3 = 0$ are

$$\theta_f = \theta_m, \ s_f = s_m, \ \frac{k_f}{\lambda_f} \frac{\partial \theta_f}{\partial x_3} = k_m \frac{\partial \theta_m}{\partial x_3}, \ M_f \frac{\partial s_f}{\partial x_3} = M_m \frac{\partial s_m}{\partial x_3}, \ \omega_f = \omega_m,$$

$$-p_f + 2\mu \frac{\partial \omega_f}{\partial x_3} = -p_m, \ h_f = h_m, \ \frac{k_f}{\lambda_f} \frac{\partial h_f}{\partial x_3} = \frac{k_m}{\lambda_m} \frac{\partial h_m}{\partial x_3},$$

$$\frac{\partial \nu_f}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{K}} (u_f - u_m), \ \frac{\partial \omega_f}{\partial x_3} = \frac{\alpha_{BJ}}{\sqrt{K}} (\nu_f - \nu_m),$$

and the boundary conditions on the lower boundary $x_3 = -d_m$

$$\omega_m(-d_m) = 0, \ \theta_m(-d_m) = 0, \ s_m(-d_m) = 0, \ \mu_{mm} H_i \text{ is continuous.}$$

5 NON-DIMENSIONALIZATION

We now non-dimensionalize the equations (30 – 33) and (34 – 37) by using the relations

$$x = d_f \hat{x}_f, \ t = \frac{d_f^2}{\lambda_f} \hat{t}_f, \ \nu_f = \frac{\nu}{d_f} \hat{E}_f, \ \theta_f = \frac{|T_0 - T_u|}{\lambda_f} \hat{\theta}_f,$$

$$s_f = \frac{|S_0 - S_u|}{M_f} \hat{s}_f, \ p_f = \frac{\rho_0 \nu^2}{d_f^2} \hat{\rho}_f, \ \hat{h}_f = \frac{H_f \lambda_f}{\eta_f} \hat{h}_f.$$ (45)

For the fluid layer, and using the relations

$$x = d_m \hat{x}_m, \ t = \frac{d_m^2}{\lambda_m} \hat{t}_m, \ \nu_m = \frac{\nu}{d_m} \hat{E}_m, \ \theta_m = \frac{|T_1 - T_0|}{\lambda_m} \hat{\theta}_m,$$

$$s_m = \frac{|S_1 - S_0|}{M_m} \hat{s}_m, \ p_m = \frac{\rho_0 \nu^2}{d_m^2} \hat{\rho}_m, \ \hat{h}_m = \frac{H_m \lambda_m}{\eta_m} \hat{h}_m.$$ (46)

For the porous medium layer. Where $\lambda_f = \frac{k_f}{(\rho c)_f}$ and $\lambda_m = \frac{k_m}{(\rho c)_m}$ are the thermal diffusivity of the fluid layer and porous medium layer respectively.

The equations (30 – 33) can be written in the form

$$P_{rf}^{-1} \frac{\partial \hat{\theta}_f}{\partial \hat{t}_f} + \hat{t}_f \cdot \nabla \hat{\theta}_f = -\nabla \hat{\rho}_f + \nabla^2 \hat{\rho}_f + R_a f \hat{\theta}_f \ e_3 - R_a s_f \ \hat{s}_f \ e_3$$

$$+ \ P_{rf}^{-1} Q_f \ \frac{\partial \hat{h}_f}{\partial \hat{x}_3} + Q_f P_{rf}^{-1} P_{mf}^{-1} (\hat{h}_f \cdot \nabla \hat{h}_f),$$

$$\frac{\partial \hat{\theta}_f}{\partial \hat{t}_f} + P_{rf} \hat{t}_f \cdot \nabla \hat{\theta}_f = \text{sign}(T_0 - T_u) \ \omega_f + \nabla^2 \hat{\theta}_f,$$ (48)

$$
\frac{1}{Le_f} \frac{\partial \hat{s}_f}{\partial \hat{t}_f} + S_c f \hat{t}_f \cdot \nabla \hat{s}_f = \text{sign}(S_0 - S_u) \ \omega_f + \nabla^2 \hat{s}_f,$$ (49)

$$\frac{\partial \hat{h}_f}{\partial \hat{t}_f} = P_{mf} \frac{\partial \hat{t}_f}{\partial \hat{x}_3} + (\hat{h}_f \cdot \nabla) \hat{E}_f - (\hat{t}_f \cdot \nabla) \hat{h}_f + P_{mf} \nabla^2 \hat{h}_f,$$ (50)

where $P_{rf}$, $P_{mf}$, $Q_{f}$, $S_{ef}$, $Le_{f}$, $Ra_{f}$ and $Ra_{sf}$ are non-dimensional numbers denote the viscous Prandtl number, magnetic Prandtl number, Chandrasekhar number, Schmidt number, Lewis number, thermal Rayleigh number and solute Rayleigh number of the fluid layer where

$$
P_{rf} = \frac{\nu}{\lambda_{f}}, \quad P_{mf} = \frac{\eta_{f}}{\lambda_{f}}, \quad Q_{f} = \frac{\mu_{mf} H_{f}^{2} d_{f}^{2}}{4 \pi \rho_{0} \nu \eta_{f}}, \quad S_{ef} = \frac{\nu}{M_{f}}, \quad Le_{f} = \frac{M_{f}}{\lambda_{f}},$$

$$
Ra_{f} = \frac{\alpha g d_{f}^{2} |T_{0} - T_{u}|}{\lambda_{f} \nu}, \quad Ra_{sf} = \frac{\alpha d_{f}^{2} |S_{0} - S_{u}|}{M_{f} \nu},
$$

the equations (34–37) can be written in the form

$$
P_{rm}^{-1} Da \frac{\partial \tilde{\nu}_{m}}{\partial x_{m}} = -\nabla \tilde{p}_{m} - \tilde{\nu}_{m} + Ra_{m} \hat{\theta}_{m} \hat{e}_{3} - Ra_{sm} \hat{s}_{m} \hat{e}_{3}
$$

$$
+ Q_{m} P_{rm}^{-1} Da \frac{\partial \tilde{h}_{m}}{\partial x_{3}} + Q_{m} Da P_{rm}^{-1} \tilde{h}_{m} (\hat{\nu}_{m} \cdot \nabla \tilde{h}_{m}),
$$

(51)

$$
G_{m} \frac{\partial \theta_{m}}{\partial x_{m}} + P_{rm} \tilde{\nu}_{m} \cdot \nabla \theta_{m} = \text{sign}(T_{1} - T_{0}) \omega_{m} + \nabla^{2} \theta_{m},
$$

(52)

$$
\frac{\varphi}{Le_{m}} \frac{\partial \hat{s}_{m}}{\partial x_{m}} + S_{cm} \tilde{\nu}_{m} \cdot \nabla \hat{s}_{m} = \text{sign}(S_{0} - S_{u}) \omega_{m} + \nabla^{2} \hat{s}_{m},
$$

(53)

$$
\frac{\partial^{2} \tilde{h}_{m}}{\partial x_{m}^{2}} = P_{mm} \frac{\partial \tilde{\nu}_{m}}{\partial x_{3}} + (\hat{h}_{m} \cdot \nabla) \tilde{\nu}_{m} - (\tilde{\nu}_{m} \cdot \nabla) \hat{h}_{m} + P_{mm} \nabla^{2} \hat{h}_{m},
$$

(54)

where $G_{m} = \frac{\rho_{f} c_{mf}}{\mu_{mf} \nu}$, $P_{rm}$, $P_{mm}$, $Q_{m}$, $S_{cm}$, $Le_{m}$, $Da$, $Ra_{m}$ and $Ra_{sm}$ are non-dimensional numbers denote the viscous Prandtl number, magnetic Prandtl number, Chandrasekhar number, Schmidt number, Lewis number, Darcy number, thermal Rayleigh number and solute Rayleigh number of the porous medium layer where

$$
P_{rm} = \frac{\nu}{\lambda_{m}}, \quad P_{mm} = \frac{\eta_{m}}{\lambda_{m}}, \quad Q_{m} = \frac{\mu_{mm} H_{m}^{2} d_{m}^{2}}{4 \pi \rho_{0} \nu \eta_{m}}, \quad S_{cm} = \frac{\nu}{M_{m}}, \quad Le_{m} = \frac{M_{m}}{\lambda_{m}},$$

$$
Da = \frac{K}{d_{m}^{2}}, \quad Ra_{m} = \frac{\alpha g d_{m}^{2} |T_{0} - T_{u}| K}{\lambda_{m} \nu}, \quad Ra_{sm} = \frac{\alpha d_{m}^{2} |S_{0} - S_{u}| K}{M_{m} \nu},
$$

Using (45) and (46) in the upper boundary at $x_{3} = 1 + f(t + x_{0})$ we obtain

$$
\text{sign}(T_{0} - T_{u}) (1 - \eta_{3}) + P_{rf} \eta_{4} \hat{\theta}_{s} + Nu_{4} (P_{rf} \hat{\theta}_{f} - \text{sign}(T_{0} - T_{u}) f) = 0,
$$

(55)

$$
\text{sign}(S_{0} - S_{u}) (1 - \eta_{4}) + S_{sf} \eta_{5} \hat{s}_{s} + Nu_{5} (S_{sf} \hat{s}_{f} - \text{sign}(S_{0} - S_{u}) f) = 0,
$$

(56)

$$
\tilde{\nu}_{s} - \frac{\partial f}{\partial x_{3}} \hat{\nu}_{s} = P_{rf}^{-1} \frac{\partial f}{\partial t},
$$

(57)

$$
\lim_{x_{3} \to 1^{-}} \hat{h}_{s} = \frac{\mu_{0}}{\mu_{mf}} \lim_{x_{3} \to 1^{+}} \frac{\partial \phi_{h}}{\partial x_{3}},
$$

(58)

$$
Cr^{-1} \quad P_{rf}^{-1} \quad b_{0} = -\tilde{p}_{f} + \frac{1}{2} Q_{f} P_{rf}^{-1} [P_{rf}^{-1} (\hat{h}_{f} \eta_{3})^{2} + 2 \nu_{j} \hat{h}_{f} \eta_{j}] + B_{0} C r^{-1} P_{rf}^{-1} f - \frac{1}{2} Q_{f} P_{mf}(1 - n_{3}^{2}) + 2 \nu_{j} \eta_{3} n_{j},
$$

(59)

$$
- \frac{1}{2} \left[ \text{sign}(T_{0} - T_{u}) P_{rf}^{-1} Ra_{f} - \text{sign}(S_{0} - S_{u}) S_{ef}^{-1} Ra_{sf} Le_{f} \right] f^{2},
$$

where $\phi_0$ is the non-dimensional magnetic potential function in the region $z_3 > 1 + f$ and $Cr$, $B_0$, $Ma$ & $Ma_0$ are non-dimensional numbers denote the Crispation number, Bond number, thermal Marangoni number and solute Marangoni number where

$$Cr = \frac{\rho_0 \nu \lambda_f}{d_f^2 \tau(T_u, S_u)}, \quad B_0 = \frac{\rho_0 \nu d_f^2}{\tau(T_u, S_u)},$$

$$Ma = -\frac{\partial \tau}{\partial T} \frac{d_f \left| T_0 - T_u \right|}{\rho_0 \nu \lambda_f}, \quad Ma_s = -\frac{\sigma_f}{\rho_0 \nu M_f}.$$

The middle boundary at $z_3 = 0$

$$\gamma_T \hat{\delta}_T(0) = \varepsilon_T \delta_m(0), \quad \gamma_S \hat{s}_S(0) = \varepsilon_S \varepsilon_m(0), \quad \frac{\partial \hat{\delta}_T(0)}{\partial \hat{x}_3} = \varepsilon_T \frac{\partial \hat{\theta}_m(0)}{\partial \hat{x}_3},$$

$$\frac{\partial \hat{s}_S(0)}{\partial \hat{x}_3} = \varepsilon_S \frac{\partial \hat{e}_m(0)}{\partial \hat{x}_3}, \quad \omega_f(0) = \hat{\omega}_m(0), \quad \hat{\phi}_f(0) - 2 \frac{\partial \hat{\phi}_f(0)}{\partial \hat{x}_3} = \frac{\hat{\lambda}^2}{Da} \hat{\tau}_m(0),$$

$$\hat{h}_f(0) = \frac{1}{\hat{\eta}_T} \hat{h}_m(0), \quad \frac{\partial \hat{h}_f(0)}{\partial \hat{x}_3} = \frac{\hat{\lambda}}{\hat{\eta}_T} \frac{\partial \hat{\tau}_m(0)}{\partial \hat{x}_3},$$

$$\frac{\partial \hat{\psi}_f(0)}{\partial \hat{x}_3} = \frac{\hat{\lambda}^2 \sigma_{BL}}{\sqrt{Da}} \left( \frac{1}{\hat{\alpha}_f} \hat{\psi}_f(0) - \hat{\psi}_m(0) \right),$$

where

$$\hat{\lambda} = \frac{d_f}{d_m}, \quad \varepsilon_T = \frac{\lambda_f}{\lambda_m}, \quad \varepsilon_S = \frac{M_f}{M_m}, \quad \hat{\eta} = \frac{H_f \eta_m}{H_m \eta_f},$$

$$\gamma_T = \frac{|T_0 - T_u|}{|T - T_0|} = \frac{\hat{\lambda}}{\varepsilon_T}, \quad \gamma_S = \frac{|S_0 - S_u|}{|S - S_0|} = \frac{\hat{\lambda}}{\varepsilon_S}, \quad P_f = \frac{1}{\varepsilon_T} P_{tm},$$

$$Ra_f = \frac{\hat{\lambda}^3}{\varepsilon_T^2 Da} Ra_m, \quad Ra_{sf} = \frac{\hat{\lambda}^3}{\varepsilon_S^2 Da} Ra_{sm}, \quad Le_f = \frac{\varepsilon_S}{\varepsilon_T} Le_m.$$

The lower boundary at $z_3 = -1$ the condition becomes

$$\omega_f = 0, \quad \delta_f = 0, \quad s_S = 0, \quad \lim_{z_3 \to -1^+} \hat{h}_m = \frac{\mu_0}{\mu_{nm}}, \quad \lim_{z_3 \to -1^+} \frac{\partial \hat{\delta}_T}{\partial \hat{x}_3} = \frac{\partial \hat{\delta}_f}{\partial \hat{x}_3}$$

where $\hat{\phi}_f$ is the non-dimensional magnetic potential function.

## 6 Linearized Problem

The linearized of equations is obtained by ignoring all products, the powers more than first and by dropping hat the superscript ( ) for fluid layer the resulting

$$P_f \frac{\partial \delta_f}{\partial z_f} = -\nabla p_f + \nabla^2 \psi_f + Ra_f \delta_f \bar{e}_3 - Ra_{sf} s_f \bar{e}_3 + Q_f \frac{1}{P_f} \frac{\partial \hat{h}_{sf}}{\partial \bar{e}_3},$$

\[
\frac{\partial \theta_f}{\partial t_f} = \nabla^2 \theta_f + \xi_T \omega_f, \quad (64)
\]
\[
\frac{1}{Le_f} \frac{\partial s_f}{\partial t_f} = \nabla^2 s_f + \xi_s \omega_f, \quad (65)
\]
\[
P_m^{-1} \frac{\partial h_f}{\partial t_f} = \frac{\partial v_f}{\partial x_3} + \nabla^2 h_f, \quad (66)
\]

and for the porous medium layer the resulting
\[
\frac{Da}{\varphi P_{rm}} \frac{\partial \mu_m}{\partial t_m} = -\nabla \mu_m - \nabla \theta_m + Ra_m \theta_m \xi_0 - Ra_{sm} s_m \xi_3 + Q_m P_{rm}^{-1} Da \frac{\partial h_m}{\partial x_3}, \quad (67)
\]
\[
G_m \frac{\partial \theta_m}{\partial t_m} = \nabla^2 \theta_m + \xi_T \omega_m, \quad (68)
\]
\[
\frac{\varphi}{Le_m} \frac{\partial s_m}{\partial t_m} = \nabla^2 s_m + \xi_s \omega_m, \quad (69)
\]
\[
P_{m,1}^{-1} \frac{\partial h_m}{\partial t_m} = \frac{\partial \mu_m}{\partial x_3} + \nabla^2 h_m, \quad (70)
\]

where
\[
\xi_T = \text{sign}(T_i - T_0) = \text{sign}(T_0 - T_u) = \begin{cases} +1 & \text{when heating from below,} \\ -1 & \text{when heating from above.} \end{cases}
\]
\[
\xi_s = \text{sign}(S_i - S_0) = \text{sign}(S_0 - S_u) = \begin{cases} +1 & \text{when soluting from below,} \\ -1 & \text{when soluting from above.} \end{cases}
\]

The boundary conditions on the lower boundary \(x_3 = -1\) are unchanged except to clear superscript, but the linearized upper boundary conditions
\[
P_{rf} \frac{\partial \theta_f}{\partial x_3} + Nu(P_{rf} \theta_f - \xi_T f) = 0,
\]
\[
S_{ef} \frac{\partial s_f}{\partial x_3} + Nu_s(S_{ef} s_f - \xi_s f) = 0,
\]
\[
\nu_3 = P^{-1}_{rf} \frac{\partial f}{\partial t_f},
\]
\[
\lim_{x_3 \to -1^-} h_i = \mu_0 \left( \lim_{x_3 \to -1^+} \frac{\partial \phi_u}{\partial x_i} \right),
\]
\[
Cr^{-1} P^{-1}_{rf} b^\alpha_a = -q_f + Q_f P^{-1}_{rf} h_3 + B_0 Cr^{-1} P^{-1}_{rf} f + \frac{2}{x_3} \frac{\partial \nu_3}{\partial x_3},
\]
\[
-M a P^{-1}_{rf} (P_{rf} \theta_{\alpha} - \xi_T f_{\alpha}) - Ma S_{ef}^{-1} (S_{ef} s_{\alpha} - \xi_s f_{\alpha}) = \nu_{3,\alpha} + \frac{\partial \nu_{\alpha}}{\partial x_3} + Q_f h_{\alpha}.
\]

To simplify the normal stress boundary condition on the interface plane we will eliminate hydrostatic pressure term, then by taking two-dimensional Laplacian of (61)_6 we obtain
\[
\Delta_2 p_f(0) - 2 \frac{\partial}{\partial x_3} \Delta_2 \omega_f(0) = \frac{\dot{\alpha}}{Da} \Delta_2 p_m(0), \quad (71)
\]

since
\[
\nabla \nu_f = 0 \implies \frac{\partial u_f}{\partial x_1} + \frac{\partial \nu_f}{\partial x_2} = -\frac{\partial \omega_f}{\partial x_3},
\]
\[
\nabla \nu_m = 0 \implies \frac{\partial u_m}{\partial x_1} + \frac{\partial \nu_m}{\partial x_2} = -\frac{\partial \omega_m}{\partial x_3}.
\]

(72)

Then we take the divergence of equations (63) and (67) we get respectively
\[
\Delta p_f = \frac{1}{P_f} \frac{\partial}{\partial t} \frac{\partial \omega_f}{\partial x_3} - \nabla^2 \frac{\partial \omega_f}{\partial x_3},
\]

(73)
\[
\Delta p_m = \frac{D_a}{\varphi P_m} \frac{\partial}{\partial t} \frac{\partial \omega_m}{\partial x_3} + \frac{\partial \omega_m}{\partial x_3},
\]

(74)

substitute from (74) and (73) into (71) to get
\[
\frac{\partial}{\partial x_3} \left( \nabla^2 \omega_f(0) - \frac{1}{P_f} \frac{\partial \omega_f(0)}{\partial t} + 2\Delta \omega_f(0) \right) = -\frac{d^t}{d a} \left( \frac{D_a}{\varphi P_m} \frac{\partial}{\partial t} + 1 \right) \frac{\partial \omega_m(0)}{\partial x_3}.
\]

(75)

Combine the Beavers and Joseph boundary conditions on the interface plane (61)_9 and (61)_10 by differentiating equations (61)_9 and (61)_10 with respect to x_1 and x_2 respectively, we get
\[
\frac{\partial}{\partial x_3} \frac{\partial u_f(0)}{\partial x_1} = \frac{\partial}{\partial x_3} \left( \frac{\partial u_f(0)}{\partial x_1} - \frac{\partial}{\partial x_1} \frac{\partial u_m(0)}{\partial x_1} \right),
\]

(76)
\[
\frac{\partial}{\partial x_3} \frac{\partial u_f(0)}{\partial x_2} = \frac{\partial}{\partial x_3} \left( \frac{\partial u_f(0)}{\partial x_2} - \frac{\partial}{\partial x_2} \frac{\partial u_m(0)}{\partial x_2} \right),
\]

(77)

by adding (76) to (77) and using (72) we get
\[
\frac{\partial}{\partial x_3} \left( \frac{\alpha_B}{\sqrt{D a}} \frac{\partial \omega_f(0)}{\partial x_3} - \frac{\partial \omega_f(0)}{\partial x_3} \right) = \frac{d^t}{\sqrt{D a}} \frac{\partial \omega_m}{\partial x_3}.
\]

(78)

Recall that the magnetic field on insulating material is irrotational and is derived from a potential function \( \phi(t, x_3) \) which is the solution of Laplace's equation. Let \( \phi = \psi(t, x_3) e^{i(p_3 + q_3 x_3)} \) then
\[
\frac{\partial^2 \psi}{\partial x_3^2} - a^2 \psi = 0, \quad a^2 = p^2 + q^2, \quad \frac{\partial \psi}{\partial x_3} \to 0 \text{ as } |x_3| \to \infty,
\]

where \( a = \sqrt{p^2 + q^2} \) the dimensionless wave number. Trivially \( \phi_t \) and \( \phi_u \) have functional form
\[
\phi_1 = C_1(t) e^{a x_3} e^{i(p x_3 + q z_3)}, \quad \phi_u = C_u(t) e^{-a x_3} e^{i(p x_3 + q z_3)}.
\]

When \( x_3 < -1 \) then \( h = C_1(t) \) (i.e., \( a, q, a \) and continuity of the magnetic induction across \( x_3 = -1 \) requires that
\[
h_{\alpha 3} = \frac{1}{\mu_n} b_{\alpha 3} = -\frac{1}{\mu_n} b_{\alpha 3} = \frac{\mu_0}{\mu_n} a^2 C_1(t) = \frac{a}{\mu_n} b_3 = a h_3.
\]

With a similar argument on \( x_3 = 1 + f(t, x_3) \). Hence the magnetic boundary conditions are
\[
\frac{\partial h_3}{\partial x_3} - a_m h_3 = 0, \quad x_3 = -1,
\]

\[
\frac{\partial h_s}{\partial x_3} + a_f h_3 = 0, \quad x_3 = 1.
\]

We put \( \omega = \nu_3, \ h = h_3 \) and take the double curl of equation (63) and (67) to eliminate hydrostatic pressure \( p_f \) and \( p_m \) respectively, and we take the third component for all equations of the system. Therefore, for the fluid layer, we get

\[
P_r^{-1} \frac{\partial}{\partial t} \nabla^2 \omega_f - Q_f P_r^{-1} \nabla^2 D h_f = \nabla^4 \omega_f + Ra_f \Delta_2 \theta_f - Ra_{sf} \Delta_2 s_f,
\]

\[
\frac{\partial \theta_f}{\partial t} = \xi_T \omega_f + \nabla^2 \theta_f,
\]

\[
\frac{1}{L_f} \frac{\partial s_f}{\partial t} = \xi_s \omega_f + \nabla^2 s_f,
\]

\[
P_m^{-1} \frac{\partial h_f}{\partial t} = \nabla^2 h_f + D \omega_f.
\]

Also for the porous medium layer, we obtain

\[
\frac{Da}{\varphi P_r m} \frac{\partial}{\partial t} \nabla^2 \omega_m - Q_m P_r^{-1} Da \nabla^2 D h_m = -\nabla^2 \omega_m + Ra_m \Delta_2 \theta_m - Ra_{sm} \Delta_2 s_m,
\]

\[
G_m \frac{\partial \theta_m}{\partial t} = \xi_T \omega_m + \nabla^2 \theta_m,
\]

\[
\frac{\varphi}{L_m} \frac{\partial s_m}{\partial t} = \xi_s \omega_m + \nabla^2 s_m,
\]

\[
P_m^{-1} \frac{\partial h_m}{\partial t} = \nabla^2 h_m + D \omega_m,
\]

where \( \nabla^2 \) is the 3-D Laplacian and \( \Delta_2 \) is the 2-D Laplacian.

Now we look for solution of the form

\[
\phi(t, x_3) = \phi(x_3) e^{\sigma t} e^{i(\varphi x_1 + \varphi x_2)}, \quad \phi = \{ \omega, \ h, \ p, \ \theta, \ s \}.
\]

\[
f(t, x_3) = f_0 e^{\sigma t} e^{i(\varphi x_1 + \varphi x_2)}, \quad f_0 \text{ is constant.}
\]

Thus the equation of fluid layer become

\[
\begin{align*}
\sigma_f P_r^{-1} (D_f^2 - a_f^2) \omega_f &= \sigma_f P_r^{-1} Q_f P_r^{-1} D_f^2 h_f = (D_f^2 - a_f^2)^2 \omega_f \\
&= Ra_f a_f^2 \theta_f + Ra_{sf} a_f^2 s_f - Q_f P_r^{-1} D_f^2 \omega_f,
\end{align*}
\]

\[
\sigma_f \theta_f = \xi_T \omega_f + (D_f^2 - a_f^2) \theta_f,
\]

\[
\frac{\sigma_f}{L_f} \theta_f = \xi_s \omega_f + (D_f^2 - a_f^2) s_f,
\]

\[
\sigma_f P_m^{-1} h_f = (D_f^2 - a_f^2) h_f + D_f \omega_f,
\]

while the porous medium equations take the form

\[
\begin{align*}
\frac{-Da \sigma_m}{\varphi P_r m} (D_m^2 - a_m^2) \omega_m - \sigma_m Da P_m^{-1} Q_m P_m^{-1} D_m h_m &= (D_m^2 - a_m^2) \omega_m \\
&+ Ra_m a_m^2 \theta_m - Ra_{sm} a_m^2 s_m + Q_m P_m^{-1} Da D_m^2 \omega_m
\end{align*}
\]

\[ G_m \sigma_m \theta_m = \xi_T \omega_m + (D_m^2 - \alpha_m^2) \sigma_m, \quad (84) \]
\[ \frac{\phi}{\sigma_m} a_m = \xi_S \omega_m + (D_m^2 - \alpha_m^2) \sigma_m, \quad (85) \]
\[ \sigma_m P_m^{-1} h_m = (D_m^2 - \alpha_m^2) h_m + D_m \omega_m, \quad (86) \]
from these equations
\[ a_f = \hat{a} a_m, \quad \sigma_f = \frac{D^2}{\varepsilon_T} \sigma_m, \]
\[ D_m = \frac{\partial}{\partial x_3}; \quad x_3 \in [-1, 0], \quad D_f = \frac{\partial}{\partial x_3}; \quad x_3 \in [0, 1], \]
where \( a_f \) and \( a_m \) are non-dimensional wave number in the fluid layer and porous medium layer respectively, \( \sigma_f \) and \( \sigma_m \) are growth rates.

As a preamble to the formulation of the final boundary conditions on \( x_3 = 1 \), the pressure everywhere is given by the equation
\[ p = \frac{1}{\alpha^2} \left( D^2 \omega - \alpha^2 D \omega + Q P_r^{-1} D^2 h - \sigma P_r^{-1} D \omega \right) \]
hence the boundary conditions on \( x_3 = 1 \) are
\[ P_r f D_f \theta_f + Nu(P_r f \theta_f - \xi_T f_0) = 0, \quad (87) \]
\[ S_{ef} D_f s_f + Nu_{e}(S_{ef} s_f - \xi_S f_0) = 0, \quad (88) \]
\[ \omega_f - \sigma_f P_r^{-1} f_0 = 0, \quad (89) \]
\[ D_f h_f + a_f h_f = 0, \quad (90) \]
\[ a_f^2 (B_0 + a_f^2) f_0 - Cr P_r f (D_f^2 \omega_f - 3a_f^2 D_f \omega_f - Q_f P_r^{-1} D_f \omega_f) = \sigma_f Cr (Q_f P_r^{-1} h_f - D_f \omega_f), \quad (91) \]
\[ (D_f^2 + a_f^2) \omega_f + Ma P_r^{-1} (P_r f \theta_f - \xi_T f_0) a_f^2 \]
\[ + Ma_{e} S_{ef}^{-1} (S_{ef} s_f - \xi_S f_0) a_f^2 = 0. \quad (92) \]
The middle boundary on \( x_3 = 0, \) become
\[ \tau_T \theta_f = \varepsilon_T \theta_m, \quad \tau_S s_f = \varepsilon_S s_m, \quad D_f \theta_f = \varepsilon_T D_m \theta_m, \quad D_f s_f = \varepsilon_S D_m s_m, \]
\[ \omega_f = \hat{a} \omega_m, \quad (93) \]
\[ D_f^3 \omega_f - 3a_f^2 D_f \omega_f - \frac{\sigma_f}{P_r f} D_f \omega_f = \frac{\hat{a}^2}{D_T} \frac{D a}{\phi P_r m} \sigma_m + 1 \right] D_m \omega_m, \]
\[ h_f = \frac{1}{\hat{a}} \omega_m, \quad (94) \]
\[ D_f h_f = \frac{\hat{a}}{\hat{a} \varepsilon_T} D_m h_m, \quad \frac{\alpha_{BF}}{\sqrt{D_T}} \hat{a} D_f \omega_f - D_f^2 \omega_f = \frac{\hat{a}^3}{\sqrt{D_T}} \alpha_{BF} D_m \omega_m. \]

The lower boundary on \( x_3 = -1 \) become
\[ \omega_m = 0, \quad \theta_m = 0, \quad s_m = 0, \quad D_m h_m - a_m h_m = 0. \quad (94) \]
7 METHOD OF SOLUTION

The first order Chebyshev spectral method is applied to solve the equations (79 – 86) with the relevant boundary conditions (87 – 94), and we map \( x_3 \in [0, 1] \) and \( x_3 \in [-1, 0] \) in to \( z \in [-1, 1] \) by the transformations \( z = 2x_3 - 1 \) and \( z = 2x_3 + 1 \) respectively, to obtain

\[
\frac{\partial}{\partial x_3} = 2 \frac{\partial}{\partial z}, \quad \text{thus} \quad D_f = D_m = 2 \frac{\partial}{\partial z} = D, \quad z \in [-1, 1].
\]

Let the variables \( y_1, y_2, \ldots, y_{18} \) be defined by

\[
\begin{align*}
y_1 &= \omega_f, & y_2 &= D_f \omega_f, & y_3 &= D_f^2 \omega_f, & y_4 &= D_f^3 \omega_f, & y_5 &= \theta_f, \\
y_6 &= D_f \theta_f, & y_7 &= \varepsilon_f, & y_8 &= D_f \varepsilon_f, & y_9 &= \varphi_f, & y_{10} &= D_f \varphi_f, \\
y_{11} &= \omega_m, & y_{12} &= D_m \omega_m, & y_{13} &= \theta_m, & y_{14} &= D_m \theta_m, & y_{15} &= \varphi_m, \\
y_{16} &= D_m \varphi_m, & y_{17} &= \varepsilon_m, & y_{18} &= D_m \varepsilon_m.
\end{align*}
\]

Then the equations (79 – 86) can be rewritten in a system of eighteen ordinary differential equations of first order, since \( D_f = D_m = D \) and if we put \( \sigma_m = \sigma \) then \( \sigma_f = \frac{\varphi_f}{\varepsilon_f} \sigma \) so that the Eigenvalue problem can be reformulated in the form

\[
\frac{dY}{dz} = AY + \sigma BY, \quad z \in [-1, 1]
\]

where \( A \) and \( B \) are real \( 18 \times 18 \) matrices. The final Eigenvalue problem reduces to \( EX = \sigma FX \) where matrices \( E \) and \( F \) have the block forms. The boundary conditions replace the \( M \)th, \( 2M \)th, \ldots, \( 18M \)th rows of \( E \) and \( F \). First to make the boundary condition on \( x_3 = 1 \) four condition. Its convenient to neglecting \( f_0 \) and neglecting normal stress upper boundary. All boundary condition take the form

\[
\begin{align*}
y_6 + Nu y_5 &= 0, \\
y_8 + Nu y_7 &= 0, \\
y_1 &= 0, \\
y_3 + \alpha_f^3 y_1 + \alpha_f^3 M a y_6 + \alpha_f^2 M a_s y_7 &= 0, \\
y_{10} + \alpha_f y_9 &= 0,
\end{align*}
\]

\[
\begin{align*}
y_1 - \frac{\alpha_f}{\varepsilon_f} y_{11} &= 0, & T_T y_5 - \varepsilon_f y_{13} &= 0, & T_S y_7 - \varepsilon_f y_{15} &= 0, & y_6 - \varepsilon_T y_{14}, \\
y_8 - \varepsilon_S y_{16} &= 0, & y_4 - 3\alpha_f^3 y_2 + \frac{\alpha_f}{D a} y_{12} &= \sigma \left( \frac{d^2}{\varepsilon_T P_f} y_2 - \frac{d^2}{\varphi P_m} y_{12} \right), \\
y_9 - \frac{1}{\mu} \frac{\alpha_f}{\varepsilon_f} y_{17} &= 0, & y_{10} - \frac{\alpha_f}{\varepsilon_f} y_{17} &= 0, & \frac{\alpha_E}{\sqrt{D a}} \frac{\alpha_f}{\varepsilon_f} y_2 - y_3 - \frac{\alpha_f}{\sqrt{D a}} \frac{\alpha_f}{\varphi} y_{12} &= 0, \\
y_{11} &= 0, & y_{13} &= 0, & y_{15} &= 0, & y_{18} - \alpha_m y_{17} &= 0.
\end{align*}
\]

Figure 2: The relation between thermal Rayleigh number in porous layer $Ra_m$ and wave number in porous layer $a_m$ for various of depth ratio when, $Da = 4 \times 10^{-6}$, $Ma = 100$, $Ma_t = 10$, $Q_s = 10$, $Ra_{m,t} = 50$.

8 RESULTS AND DISCUSSION

The Eigenvalue problem (79 – 86) with boundary conditions (87 – 94) by using first order of Chebyshev spectral method is transformed to a system of ten ordinary differential equations of first order in the fluid layer and a system of eight ordinary differential equations of first order in the porous layer with eighteen boundary conditions. In this paper we will discuss the numerical results when the Eigenvalues are real then the stationary instability happens as shown in the figures (2) – (11). These figures display the relation between $Ra_m$ and $a_m$ by considering chosen values of the non-dimensions constants

$P_{rf} = 6$, $G_m = 10$, $\varphi = 0.3$, $\alpha_{BJ} = 0.1$, $Le_f = 1$, $P_{mm} = 6$, $\varepsilon_T = 0.7$, $\varepsilon_S = 3.75$, $P_{mf} = 6$, $\hat{n} = 1$, $Nu_t = 10$, $Nu_s = 5$, $\xi_T = 1$, $\xi_S = 1$.

For deferent values of $\hat{d}$, $Q_m$, $Da$, $Ma$, $Ra_a$, $Ra_{m,t}$ we found the following results will be held:

1. The increasing of depth ratio has unstablizing effect for the system as shown in figure (2).

2. Linear magnetic field has a stabilizing effect for the system as shown in figure (3).

3. The permeability of porous medium has a stabilizing effect for the system on the other hand the fluid becomes more stable when the permeability increases as shown in figures (4) and (5).

4. The increasing of the thermal Marangoni number has unstablizing effect for the system as shown in figures (6), (7) and (8).

Figure 3: The relation between thermal Rayleigh number in porous layer $Ra_m$ and wave number in porous layer $a_m$ for various of Chandrasekhar number when, $Da = 4 \times 10^{-6}$, $Ma = 100$, $Ma_s = 10$, $\dot{d} = 0.07$, $Ra_{m_0} = 50$.

5. The increasing of the solute Marangoni number has unstabilizing effect for the system as shown in figures (9) and (10).

6. The increasing of the solute Rayleigh number in porous layer has a stabilizing effect for the system, i.e. there is a direct relation between thermal and solute Rayleigh number in porous layer as shown in figure (11).

9 CONCLUSIONS

In this paper we have implemented a D variant of the Chebyshev numerical method and have derived accurate results for the Marangoni Convections in a horizontal porous layer superposed by a fluid layer in the presence of uniform vertical magnetic field and solute. The Chebyshev method is very important for this general class of problem since it is highly accurate, the eigenfunctions are easy to generate, and it is easily generalized to the multilayer situation where there are many porous-fluid layers superposed. The Chebyshev method used here also yields as many eigenvalues in the spectrum as one desires. This is particularly important in that for convection problems of current interest, it is often the case that the eigenvalues change position, one eigenvalues which is dominant in a certain region of parameter space being replaced by another in another parameter region. Firstly we have investigated in detail the effect of depth ratio on the onset of instability. Straughan's (2001) study the effect of surface tension is investigated in detail and it is found that for the parameter $\dot{d}$ very small, the surface tension has strong effect on convection dominated by the porous medium, whereas for $\dot{d}$ larger the surface tension effect is observed only with the fluid mode as shown in figure (12). Also Chen and Chen (1988)produced

a classical paper in which they studied thermal convection in a two-layer system composed of a porous layer saturated with fluid over which was a layer of the same fluid. The layer was heated from below and Chen and Chen (1988) considered the bottom of the porous layer, as well as the upper surface of the fluid, to be fixed. They showed that the linear instability curves for the onset of convection motion, i.e., the Rayleigh number against wavenumber curves, may be bimodal in that the curves possess two local minima. A crucial parameter in their analysis is the number $\tilde{d}$. They interpreted their finding by showing that for $\tilde{d}$ small ($\leq 0.13$) the instability was initiated in the porous medium, whereas for $\tilde{d}$ larger than this the mechanism changed and instability was controlled by the fluid layer as shown in figure (13). Al Enizi (Al Enizi 2006) study where the solute concentration effect was included, the effect depth ratio become more stronger, due to the addition of solute concentration with the effect of surface tension as shown in figure (14). In our work it is clear from the figure that the Rayleigh number in the porous layer decreases continuously as the thickness of the layer increases. And it is clear the magnetic field has a stabilizing effect on the system. Finale we can take water, oil or glycerin as example of fluid and sponge as example of porous layer then we can obtain the results.

REFERENCES


Figure 5: The relation between thermal Rayleigh number in porous layer $Ra_m$ and wave number in porous layer $a_m$ for various of Darcy number when, $Ma = 100$, $Ma_s = 10$, $d = 0.1$, $Ra_{me} = 50$, $Q_m = 10$.


Figure 6: The relation between thermal Rayleigh number in porous layer $Ra_m$ and wave number in porous layer $a_m$ for various of thermal Marangoni number when, $Da = 4 \times 10^{-6}$, $Q_m = 10$, $Ma_s = 10$, $d = 0.05$, $Ra_{m,s} = 50$.


Figure 8: The relation between thermal Rayleigh number in porous layer $Ra_m$ and wave number in porous layer $a_m$ for various of thermal Marangoni number when, $Da = 4 \times 10^{-6}$, $Q_m = 10$, $Ma_e = 10$, $d = 0.1$, $Ra_m = 50$.


Figure 9: The relation between thermal Rayleigh number in porous layer $Ra_m$ and wave number in porous layer $a_m$ for various of solute Marangoni number when, $Da = 4 \times 10^{-6}$, $Q_m = 10$, $Ma = 100$, $\bar{d} = 0.05$, $Ra_{ms} = 50$. 

Figure 10: The relation between thermal Rayleigh number in porous layer $Ra_m$ and wave number in porous layer $a_m$ for various of solute Marangoni number when, $Da = 4 \times 10^{-6}$, $Q_m = 10$, $Ma = 100$, $d = 0.1$, $Ra_{ms} = 50$.

Figure 11: The relation between thermal Rayleigh number in porous layer $Ra_m$ and wave number in porous layer $a_m$ for various of solute Rayleigh number when, $Ma = 100$, $Ma_s = 10$, $d = 0.05$, $Da = 4 \times 10^{-6}$, $Q_m = 10$.

Figure 12: The relation between thermal Rayleigh number in porous layer and wave number in porous layer for various of depth ratio when, $Ma = -100 \phi = 0.3, Pr = 6$. 

Figure 13: The relation between thermal Rayleigh number in porous layer and wave number in porous layer for various of depth ratio.

Figure 14: The relation between thermal Rayleigh number in porous layer and wave number in porous layer for various of dept ratio when, $Ma = 100$, $Ma_s = 10$, $Da = 4 \times 10^{-6}$.