Electrorheological Kelvin–Helmholtz instability of a fluid sheet

Yusry O. El-Dib a,⁎, R.T. Matoog b

a Department of Mathematics, Faculty of Education, Ain Shams University, Heliopolis, Cairo, Egypt
b Department of Mathematics, Faculty of Applied Science, Umm Al-Qura University, Makkah, Saudi Arabia

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Abstract

The present work deals with the gravitational stability of an electrified Maxwellian fluid sheet shearing under the influence of a vertical periodic electric field. The field produces surface charges on the interfaces of the fluid sheet. Due to the rather complicated nature of the problem a mathematical simplification is considered where the weak effects of viscoelastic fluids are taken into account. The solutions of the linearized equations of motion and boundary conditions lead to two simultaneous Mathieu equations with damping terms and having complex coefficients. Stability criteria are discussed through the assumption of symmetric and anti-symmetric deformations. The disappearance of surface charges from the interfaces obeys a certain relation derived in the marginal state. Furthermore, the case dealing with general deformation is discussed through marginal state analysis. The stability behavior in resonant and nonresonant cases are studied. In addition, the stability picture in the case of absence of the field frequency is studied. The numerical examination for stability showed that the relaxation time ratio plays a destabilizing influence in the case of anti-symmetric deformation or in the general deformation. The stabilizing effect for the relaxation time ratio is saved in the case of general deformation in the presence of the field frequency. In the later case the viscosity, the velocity, and the thickness parameter play a stabilizing influence. A dual role is realized for these parameters in the absence of the field frequency or in the anti-symmetric deformation. The field frequency still plays a destabilizing role in both cases.

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1. Introduction

A host of efforts have been devoted to gaining insight into the behavior of liquids under the influence of gravity and capillary forces and subjected to electric fields. These studies were primarily motivated by numerous industrial applications, e.g., electrostatic paint spraying, image making, ink matrix printers, electrophoretic mixing, propulsion, and heat transfer. A second motivation for such studies was the need to understand and control the location and motion of liquids. Recently, there has been renewed interest in the confinement of liquids by surface tension, surface charges, and viscoelastic forces, which examines the possibility of avoiding contact between a liquid and a containing vessel for applications to material processing.

Electric fields can be described as either uniform or nonuniform, and fluids are either charged or uncharged. The movement of charged particles due to either field type is referred to as electrophoresis. The movement of uncharged particles due to a nonuniform field is known as dielectrophoresis. It is not surprising that any type of electric field can exert force on a charged particle, since the electric field itself is the result of charged particles.

Dielectrophoretic forces are usually mild, and they require relatively high nonuniform field strengths to produce any noticeable effect. A simple test to determine whether dielectrophoretic effects are at play is to switch the polarity of the electrodes or to use an ac rather than dc supply voltage. If the particles move in the same fashion for either supply voltage, the forces are most likely dielectrophoretic in nature. Fluids with low viscosity are most influenced by these forces since frictional effects are minimized. Dielectrophoretic forces are supply voltage-dependent, and the particles follow
where the viscoelastic effects are very small, so that they have no effect. The stability analyses are formulated from two perspectives. The first deals with symmetric and antisymmetric deformations. The second is concerned with general deformations. Two ordinary second-order differential equations of Mathieu type with damped terms and having complex coefficients are formulated due to the general deformation, and are analyzed by the method of multiple time-scales [26]. This technique is controlled so that the investigations will be divided into two versions according to two time scales, $T_0$ and $T_1$, which are introduced as a fast time-scale and a slow time-scale, respectively. This method is based on a smallness parameter $\varepsilon$ [27] and is applied through the assumption that the amplitude of the electric field is small [27].

The contribution of the electric field passes through the second version of the perturbations. Therefore, the resulting equations that are governing the first version (the fast solution) should not include the electric field effects. In this version, the solution analysis imposes a fourth-order dispersion relation with complex coefficients.

The set of slow-solution equations contains inhomogeneous equations. The uniform solution is required to eliminate secular terms. This elimination reduces to the well-known solvability conditions. Here, two kinds of solvability condition are available in the second-order version of the perturbation. The first is the case of the field frequency away from the disturbance frequency, which is called the nonresonant case. The second is the resonant case arising when the field frequency approaches the disturbance frequency. A second-order dispersion relation with complex coefficient is derived in the nonresonant case, while a fourth order dispersion relation is formulated in the resonant case.

2. Formulation of the problem and the fundamental equations

The derivation of the dynamical system for a viscoelastic fluid of Maxwellian type is presented in this section. Consider a horizontal fluid sheet of finite thickness $2h$, of density $\rho$, embedded in two semi-infinite fluids having densities $\rho_0$ and $\rho_3$. Both of the fluids are incompressible and isotropic with dielectric constants $\varepsilon(0), \varepsilon(2)$, and $\varepsilon(3)$. The fluids are streaming by constant velocities $v_0, v_2$, and $v_3$ along the positive x-direction. The system is assumed to be stressed by a periodic vertical electric field characterized by $E_0(0)$, $E_0(2)$, and $E_0(3)$ respectively along the negative y-direction. The superscripts (1), (2), and (3) refer to quantities in the upper fluid, plane sheet, and lower fluid, respectively. The axis $y = 0$ is taken to be the middle plane of the sheet. The effect of gravity $g$ is taken in the negative y-direction. The unperturbed fluid interfaces are parallel and the flows in each phase are everywhere parallel to each other. After small perturbation both the flat interfaces will be disturbed with two different amplitudes $\xi_1$ and $\xi_2$. A Cartesian coordinate system is considered, where the y-axis is taken vertically upward and the x-axis is taken horizontally to be at the middle sheet. The fluids have a viscoelastic nature described by the following Maxwellian constitutive relation:

$$\tau_{ij} + \lambda \left( \frac{\partial u_i}{\partial t} + \nu \frac{\partial u_i}{\partial x_k} \right) \tau_{ij} = \mu \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right),$$

(1)

where $\tau_{ij}$ is the stress tensor, $\mu$ is the coefficient of viscosity, $\lambda$ is the relaxation time, and $\nu$ is the fluid velocity vector. The total stress tensor for electroviscoelastic flow of Maxwell type is given by

$$\sigma_{ij} = -P \delta_{ij} + M_{ij} + \tau_{ij},$$

(2)

where $P$ is the hydrostatic pressure and $M_{ij}$ is the Maxwell electric stress tensor which is defined as

$$M_{ij} = \varepsilon E_i E_j - \frac{1}{2} \varepsilon E^2 \delta_{ij}.$$  

(3)

There are no electrical volume-force density terms here because $\varepsilon$ is constant in a fluid phase and there is no volume charge in the bulk of the fluids. Thus electrical forces only act on interfaces. Their contribution passes through the normal component stress term in the boundary conditions at the surface of separation. In formulating Maxwell relations for the system, we assume that the quasi-static approximation is valid for the problem [28]. With a quasi-static model, it is recognized that relevant time rates of change are sufficiently low so that contributions due to a particular dynamic process are ignorable. The objective in electrified fluids is concerned with phenomena in which electric energy storage much exceeds magnetic energy storage, and where the propagation times of electromagnetic waves are complete in shorter times compared to those of interest to us. Thus under the quasi-electrostatic approximation Maxwell’s equations are reduced to

$$\nabla \times E^{(r)} = 0,$$  

(4)

$$\nabla \cdot (\varepsilon^{(r)} E^{(r)}) = 0, \quad r = 1, 2, 3,$$  

(5)

where the electric field can be derived from a scalar function $\phi$ such that

$$E^{(r)} = -\nabla \phi^{(r)}.$$  

(6)

The potential $\phi$ has to satisfy Laplace’s equation

$$\nabla^2 \phi^{(r)} = 0.$$  

(7)

Viscoelastic fluid problems involving two nonstreaming fluids are exceedingly difficult to solve. The inclusion of external periodic forces leads to more difficulty even in the absence of relative motion between the fluids. The boundary-value problem of the Kelvin–Helmholtz model for viscous flow or for viscoelastic flow is an ill-posed problem. This is because the tangential velocity could have a non-vanishing jump. In order to modify the Kelvin–Helmholtz waves for viscoelastic flow we restrict ourselves to an approximate treatment. The assumption of weak viscoelastic forces seems to be very urgent. However, we confine the analysis here to
y = (-1)^{j+1}a + \xi_j, \quad j = 1, 2, \quad (18)

(n_0^2 - n_3^2)(\sigma_{33}^{(j)} - \sigma_{33}^{(j+1)})
+ n_0 n_3 (\sigma_{33}^{(j)} - \sigma_{33}^{(j+1)}) - (\sigma_{33}^{(j)} - \sigma_{33}^{(j+1)}) = 0,

y = (-1)^{j+1}a + \xi_j, \quad j = 1, 2. \quad (19)

(iv) Two boundary conditions account for the effect of the field on the mechanics. Consistent with the electrical laws are the continuity conditions on the tangential electric field intensity. Thus,

\[ \mathbf{n}_j \times (\mathbf{E}^{(j)} - \mathbf{E}^{(j+1)}) = 0, \]

\[ y = (-1)^{j+1}a + \xi_j, \quad j = 1, 2. \quad (20) \]

Since there are surface charges present in the equilibrium state for an isolated system, the normal electric displacement is discontinuous by the surface charge density

\[ \mathbf{n}_j \cdot (\mathbf{D}^{(j)} - \mathbf{D}^{(j+1)}) = \tilde{Q}_j, \]

\[ y = (-1)^{j+1}a + \xi_j, \quad j = 1, 2. \quad (21) \]

where \( \tilde{Q}_j \) represents the surface charge density at the interface \( j \).

The boundary conditions represented above are prescribed at the interface \( y = \pm a + \xi \). It is necessary to express all the physical quantities involved in terms of Taylor expansion about \( y = \pm a \), where \( a \) is the distance of the unperturbed interface.

### 4. Perturbation equations and linearized solutions

In what follows, the equations used to assess the linear stability of the uniform flow equations are presented. All disturbances are assumed to be two-dimensional, that is uniform across the width of the fluid sheet. The amplitude of waves that form on the fluid sheet is assumed to be small; thus, wave solutions can be obtained by assuming linear perturbations about the uniform flow solution discussed in the previous sections as

\[ \mathbf{E}^{(r)} = -E_0^{(r)} \cos \alpha \mathbf{e}_y - \nabla \phi_0^{(r)}, \quad (22) \]

\[ \mathbf{v}^{(r)} = v_0^{(r)} + \frac{\partial \psi_0^{(r)}}{\partial y} \mathbf{e}_y - \frac{\partial \phi_0^{(r)}}{\partial x} \mathbf{e}_x, \quad (23) \]

\[ p^{(r)} = p_0^{(r)} + p_1^{(r)}, \quad r = 1, 2, 3, \quad (24) \]

where \( \psi_0, \phi_0, \) and \( P_1 \) are the increments in the stream function \( \psi \), the electric potential \( \phi \), and the pressure \( P \), respectively. These expansions are introduced into the governing equations and boundary conditions, and terms quadratic or higher in the perturbed quantities are ignored; that is, a linear approximation is considered [31].

To test the stability of the present problem, the interface between the two fluids will be assumed to be perturbed about its equilibrium location and to cause displacement of the material particles of the fluid system. This displacement may be described by the equations

\[ \xi_j(x, t) = \gamma_j \psi_j(t) \exp(ikx), \quad (25) \]

where \( k \) is the wavenumber, which is assumed to be real and positive, \( i = \sqrt{-1} \), and \( \gamma_j(t) \) are arbitrary functions of time \( t \) which determine the behavior of the amplitude of the disturbance of the interfaces. The deformation in the interfaces \( \gamma = \pm a \) is due to the perturbation about the equilibrium values for all the other variables. According to linear perturbation theory [32], the unit normal vector to interfaces can be derived as

\[ \mathbf{n}_j = -\frac{\partial \tilde{Q}_j}{\partial x} \mathbf{e}_x + \mathbf{e}_y, \quad j = 1, 2. \quad (26) \]

The equations of motion and the boundary conditions will be solved for these perturbations under the assumption that the perturbations are small; that is, all equations and boundary conditions will be linearized in the perturbation quantities. Actually, the linearized equations governing the perturbation quantities are readily found to satisfy Laplace’s equations:

\[ \nabla^2 \left( \frac{\partial}{\partial t} + \frac{1}{c_r} \frac{\partial}{\partial x} \right) \phi_1^{(r)}(x, y, t) = 0, \quad (27) \]

\[ \nabla^2 \psi_1^{(r)}(x, y, t) = 0. \quad (28) \]

As a result of the perturbation and in view of (25), \( \psi_1, \phi_1 \) may assume the following dependence in the three fluid regions,

\[ \psi_1^{(r)} = \left( \frac{1}{\partial t} + ikv_0^{(r)} \right)^{-1} \psi_1^{(r)}(t, y) \exp(ikx), \]

\[ \phi_1^{(r)} = \phi_1^{(r)}(t, y) \exp(ikx), \quad (29) \]

where it is necessary to determine the unknown eigenfunctions \( \phi_1^{(1)} \) and \( \psi_1^{(1)} \). These eigenfunctions for electric potential \( \phi \) and stream function \( \psi \) die away exponentially with distance from the interfaces. However, if we substitute into Eqs. (27) and (28), the solution of the resulting differential equations leads to

\[ \psi_1^{(1)} = A_1^{(1)}(t) \exp(-ky) \exp(ikx), \quad y > a, \quad (31) \]

\[ \psi_1^{(2)} = A_1^{(2)}(t) \exp(-ky) + B_1^{(2)}(t) \exp(ky) \exp(ikx), \quad |y| < a, \quad (32) \]

\[ \psi_1^{(3)} = B_1^{(3)}(t) \exp(ky) \exp(ikx), \quad y < -a, \quad (33) \]

\[ \phi_1^{(1)} = C_1^{(1)}(t) \exp(-ky) \exp(ikx), \quad y > a, \quad (34) \]

\[ \phi_1^{(2)} = C_1^{(2)}(t) \exp(-ky) + D_1^{(2)}(t) \exp(ky) \exp(ikx), \quad |y| < a, \quad (35) \]

\[ \phi_1^{(3)} = D_1^{(3)}(t) \exp(ky) \exp(ikx), \quad y < -a, \quad (36) \]

where the operator \( (\partial/\partial t + ikv_0^{(r)}) \) has been included into the unknowns \( A_1^{(1)}(t) \) and \( B_1^{(2)}(t) \). The set of time-dependence coefficients can be evaluated by making use of the appropriate boundary conditions.
Four different cases are available in order to present the periodic solution. The vanishing of the coefficient \(\alpha_0\) is trivially satisfied for the inviscid flow. In addition, the vanishing of the velocities \(v_0^{(1)} = v_0^{(2)} = v_0^{(3)} = 0\) (Rayleigh–Taylor model) or when the system has streaming with the same velocities \(v_0^{(1)} = v_0^{(2)} = v_0^{(3)} = v_0\) leads to vanishing \(\alpha_0\).

Furthermore, when the fluid sheet is embedded between two identical media \(\rho^{(1)} = \rho^{(2)}\), \(\varepsilon^{(1)} = \varepsilon^{(2)}\), \(\mu^{(1)} = \mu^{(2)}\), \(v_0^{(1)} = v_0^{(2)}\), \(E_0^{(1)} = E_0^{(2)}\), the periodic solution arises. Otherwise, for the Kelvin–Helmholtz model, which holds for viscoelastic fluids having different velocities, the periodic solution is valid when the following condition is satisfied:

\[
\varepsilon^{(1)} E_0^{(1)} \mu^{(2)} \mu^{(3)} (v_0^{(2)} - v_0^{(3)})
+ \varepsilon^{(3)} E_0^{(3)} \mu^{(2)} \mu^{(3)} (v_0^{(1)} - v_0^{(2)})
= \varepsilon^{(2)} E_0^{(2)} \mu^{(1)} \mu^{(3)} (v_0^{(1)} - v_0^{(3)}),
\]

(46)

Combining relation (37) with (46), we obtain

\[
\varepsilon^{(1)} E_0^{(1)}
= \frac{(Q_1 + Q_2)(u_0 - 1 - u_b - Q_1 + u_0 - 1 - u_b)}{\mu_b(1 - u_b) + \mu_a(u_0 - 1 - u_b)}
\]

(47)

\[
\varepsilon^{(2)} E_0^{(3)}
= \frac{(Q_2 - Q_1 + u_b - 1 - u_b)}{\mu_b(1 - u_b) + \mu_a(u_0 - 1 - u_b)}
\]

(48)

\[
\varepsilon^{(3)} E_0^{(3)}
= \frac{(Q_2 - Q_1 + u_b - 1 - u_b)}{\mu_b(1 - u_b) + \mu_a(u_0 - 1 - u_b)}
\]

(49)

where

\[
\mu_a = \frac{\mu^{(1)}}{\mu^{(2)}}, \quad \mu_b = \frac{\mu^{(3)}}{\mu^{(2)}}, \quad v_0 = \frac{v_0^{(1)}}{v_0^{(2)}}, \quad Q_j = Q_j \sigma f
\]

(50)

is used. Here \(\sigma f\) is a dimensionless quantity that represents the amplitude of the surface charge density. It is clear that \(Q_j\) will disappear from the interfaces as

\[
\mu_b(1 - u_b) + \mu_a(u_0 - 1 - u_b) = 0.
\]

At this stage the fluid behaves as conducting fluids.

The above relations (47), (48), and (49) give the electric displacement in terms of surface charge density, viscosity ratios, and velocity ratios, while the elasticity parameters make no contribution to it. These relations hold only for different velocities and show that the Rayleigh–Taylor results [6] cannot be recovered from it. In the case \(v_0^{(1)} = v_0^{(3)} = 0\), \(v_0^{(2)} \neq 0\) the above relations become

\[
\varepsilon^{(1)} E_0^{(1)}
= \frac{(Q_1 + Q_2) \mu_a \sigma f}{\mu_a - \mu_b},
\]

(51)

\[
\varepsilon^{(2)} E_0^{(2)}
= \frac{(Q_2 - Q_1) \mu_a \sigma f}{\mu_a - \mu_b},
\]

(52)

\[
\varepsilon^{(3)} E_0^{(3)}
= \frac{(Q_1 + Q_2) \mu_b \sigma f}{\mu_a - \mu_b},
\]

(53)

which are independent of the velocity contribution. It appears from the above three relations (51), (52), and (53) that the vertical field will produce surface charges on the interfaces when \(\mu_a \neq \mu_b\). This means that the surface charges will disappear on the interfaces between the fluids when the most upper fluid and the most lower fluid have the same viscosity. However, the Mathieu equation (45), in view of condition (46) and the relations (47)–(49) produces marginal stability, which is governed by the following equation:

\[
\frac{d^2 \eta}{d \xi^2} + (\delta_0 - \beta_0^2 + q_0 \cos^2 \omega \tau) \eta = 0.
\]

(54)

This recovers the surface wave behavior under the influence of the relation between viscosity, surface charges, and velocities. The stability condition in the case of constant electric field reduces to

\[
\delta_0 - \beta_0^2 + q_0 > 0.
\]

(55)

Using relations (47)–(49) the stability condition (55) can be sought in the form

\[
\sigma_2 \varepsilon_0 = \delta_0 + \beta_0^2 > 0, \quad \text{where} \quad q_0 = \sigma_2 \varepsilon_0
\]

(56)

The transition curve separating the stable region from the unstable region corresponds to

\[
\sigma_2 = \frac{-(\delta_0 + \beta_0^2)}{q_0}
\]

(57)

In the presence of the field periodicity, we prefer to discuss the stability implications of the Mathieu equation (54) according to the Melchian [32] condition

\[
\delta_0^2 + 16(\omega^2 - \delta_0 - \beta_0^2) q_0
+ 32(\delta_0 + \beta_0^2)(\omega^2 - \delta_0 - \beta_0^2) > 0,
\]

(58)

Using relations (47)–(49) the above stability condition can be written in terms of the surface charge magnitude \(\sigma_f\) as

\[
\sigma_2^2 \varepsilon_0^2 + 16 \sigma_2 \varepsilon_0 (\omega^2 - \delta_0 - \beta_0^2)
+ 32(\delta_0 + \beta_0^2)(\omega^2 - \delta_0 - \beta_0^2) > 0,
\]

(59)

which can be rewritten in the form

\[
(\sigma_2^2 - \delta_0^2)(\sigma_2^2 - \beta_0^2) > 0.
\]

(60)

It follows that the above condition is satisfied if

\[
\sigma_2^2 > \delta_0^2 \quad \text{or} \quad \sigma_2^2 < \beta_0^2 \quad (\sigma_1^* \neq \sigma_2^*),
\]

(61)

where

\[
\sigma_1^* = \frac{8}{9} \left\{ -\left(\omega^2 - \delta_0 - \beta_0^2\right)
\pm \sqrt{\left(\omega^2 - \delta_0 - \beta_0^2\right)^2 - 2\left(\omega^2 - \delta_0 - \beta_0^2\right)} \right\}
\]

(62)

The implication of the behavior of the parameter \(\sigma_2^2\) in the interfacial stability will be discussed numerically by drawing the above transition curves.
The influence of the viscosity ratios $\mu_a$ and $\mu_b$ is as follows. As the viscosity ratios $\mu_a$ and $\mu_b$ are increased, the stable region $S_1$ is decreased until the minimum point $\mu_a = 1$ or $\mu_b = 0.4$ is arrived at. Again the stability appears as $\mu_a$ or $\mu_b$ is moved away from the minimum points. At this stage the increase in surface charges plays a stabilizing role. The stable region $S_2$ is increased until the maximum points. After this the decrease in $S_2$ begins where the dual role of increasing $\mu_a$ and $\mu_b$ is observed. The continued increase of $\mu_a$ or $\mu_b$ produces a major unstable region as observed in the graphs. However, one can conclude that both the surface charge density and the viscosity ratios $\mu_a$ and $\mu_b$ play a dual role in the stability criteria.

The influence of the sheet thickness on the stability picture is displayed in Fig. 3, for the same system as in Figs. 1 and 2, where $v_o = 1.3$ and $k$ is held fixed to $k = 0.2$. The variations of the parameter $a$ are $0.5, 1, 1.5$, and $2$. The curve labeled by the symbol $\circ$ refers to $a = 0.5$, the curve marked by the symbol $+$ represents the case of $a = 1$, the case of $a = 1.5$ is denoted by the symbol $\times$, and the case of $a = 2$ is indicated by the symbol $\cdot$. The maximum instability occurs at the point $\mu_b = 0.5$, while it is independent of the thickness $a$. It is observed that as $a$ is increased the stable region $S_1$ is increased. This means that the increase in $a$ plays a stabilizing role in the $S_1$-region. In the $S_2$-region, there is a dual effect on the stabilizing influence as $a$ is increased.

The influence of the velocity ratio $v_o$ on the stability is displayed in Fig. 4, where the graph collects four different values $v_o$. The curve for the case $v_o = 1.2$ is marked by the symbol $\circ$, the curve for the case $v_o = 1.3$ is denoted by the symbol $\times$, the curve for the case $v_o = 1.4$ is represented by the symbol $\cdot$, and the curve for the case $v_o = 1.5$ is labeled by the symbol $\ast$.

Both the minimum points and the maximum points are functions of the velocity ratios $v_o$. The contact point with the $\mu_a$- or $\mu_b$-axis can be evaluated from the relation (50). At this end, $\mu_b = 0.4$ for $v_o = 1.2$, $\mu_b = 0.5$ for $v_o = 1.3$, $\mu_b = 0.5/4 \frac{1}{2}$ for $v_o = 1.4$, and $\mu_b = 0.625$ for $v_o = 1.5$. It appears that as $v_o$ is increased the surface charge density curve moves down and shifts to the direction of increasing $\mu_a$, causing a contraction in the stable regions. Both the stable regions $S_1$ and $S_2$ decrease as $v_o$ is increased. This means that $v_o$ plays a destabilizing role.

Another stability profile will be obtained when the electrocapillary excitation is switched off and the oscillatory field excitation is switched on. Numerical calculations were made for the transition curves (61). The results are displayed in Figs. 9–10 to indicate the influence of the field frequency $\omega$, the sheet thickness $a$, the viscosity ratio $\mu_a$, the relaxation time ratio $\lambda_o$, and the velocity ratio $v_o$ on the stability picture in the presence of the surface charge density. The graphs displayed into the plane $(\sigma^2 - k)$ for the same system are considered in Fig. 1. The influence of the presence of the field frequency $\omega$ on the stability criteria shows that the plane $(\sigma^2 - k)$ has been partitioned by the transition curves $\sigma^2_1$ and $\sigma^2_2$ into stable and unstable regions. These regions are found as functions of the field frequency $\omega$, the sheet thickness $a$, the viscosity ratio $\mu_a$, the elasticity ratio $\lambda_o$, and the velocity ratio $v_o$.

The stability investigations are illustrated by three different numerical values for each parameter mentioned above. The transition curve labeled by the symbol I refers to the lower value for each parameter. The curve labeled by the symbol II represents the case of the mid-value. The curve marked by the symbol III indicates the larger value that is chosen in the numerical estimation.

In Fig. 5 we plot the transition curves $\sigma^2_1$ and $\sigma^2_2$ from the relation (62) for the same system as considered in Fig. 1. Three different values of $\omega$ are taken into account for comparison: $\omega = 9.5$, 10, and 10.5. For the specific value $\omega = 10.5$, we observe that the plane has been divided into a stable region and an unstable region. The unstable region has been
6. Stability behavior when the surface deflections are independent

To discuss the linearized equations that govern wave propagation where the surface deflections \( \xi_1 \) and \( \xi_2 \) are independent, we return to the system (38), which represents coupled Mathieu equations with damped terms. Such equations, having growth rate solutions and a stability analysis, are rather complicated. In order to relax the complexity of these equations we shall construct the stability configuration near the marginal state. Thus, we shall be dealing with the periodic solutions for these equations. To accomplish this state two conditions must be satisfied: the necessary condition and the sufficient condition. They are, respectively,

\[
\frac{d^2 \Gamma}{dr^2} + i B \frac{d \Gamma}{dr} + (C + \tilde{G} \cos^2 \omega t) \Gamma = 0, \tag{63}
\]

\[
A \frac{d \Gamma}{dr} + i D \Gamma = 0. \tag{64}
\]

It is worthwhile to observe that for the nonsecular matrix \( A \) the equation that governs the marginal state can be formulated by combining the necessary condition (63) with the sufficient condition (64) into a single condition. This can be accomplished by eliminating the damping term \( \Gamma \) between them. Thus, we obtain

\[
\frac{d^2 \Gamma}{dr^2} + (\tilde{H} + \tilde{G} \cos^2 \omega t) \Gamma = 0, \tag{65}
\]

where the matrix \( \tilde{H} \) is formulated as

\[
\tilde{H} = C + \tilde{G} \Delta^{-1} D. \tag{66}
\]

In order to discuss the influence of the surface charge amplitude \( \sigma \) on the stability configuration in the general case we may introduce the nondimensional parameter \( \tilde{E} \) such that

\[
\tilde{E}^{(1)}_0 = \sqrt{E} E^{(1)}_0, \tag{67}
\]

where \( E^{(1)}_0 \) are some finite constants having the dimension of an electric field. In view of (37) the parameter \( \tilde{E} \) may be sought in terms of the surface charge densities \( Q_1 \) and \( Q_2 \) as

\[
\tilde{E} = \frac{Q_1 Q_2 \sigma^2}{(\varepsilon^{(1)}_0 E^{(1)}_{00} - \varepsilon^{(2)} E^{(2)}_{00})(\varepsilon^{(2)} E^{(2)}_{00} - \varepsilon^{(3)} E^{(3)}_{00})}, \tag{68}
\]

In this stage the above system (65) can be rewritten in the form

\[
\frac{d^2 \Gamma}{dr^2} + (\tilde{H} + \sigma^2 \tilde{G} \cos^2 \omega t) \Gamma = 0, \tag{69}
\]

where the relation (67) is used in both \( \tilde{H} \) and \( \tilde{G} \), so that \( \tilde{H} \) and \( \tilde{G} \) are two functions of \( E^{(1)}_0 \). In addition, \( \tilde{G} \) is formulated as

\[
\tilde{G} = \frac{Q_1 Q_2 \tilde{G}^{(2)}_{00}}{(\varepsilon^{(1)}_0 E^{(1)}_{00} - \varepsilon^{(2)} E^{(2)}_{00})(\varepsilon^{(2)} E^{(2)}_{00} - \varepsilon^{(3)} E^{(3)}_{00})}. \tag{70}
\]
and
\[
\hat{B} = \left[ 2(H_{11} + H_{21}) \right] (H_{22} G_{12} + H_{12} G_{22})
- (H_{11} G_{11} + H_{21} G_{11})^2
+ 2(G_{11} + G_{21})(H_{11} H_{21} - H_{22} H_{12})
\times \left[ 4(H_{22} G_{12} + H_{12} G_{22}) - 3(H_{11} G_{21} + H_{21} G_{11})
+ (H_{11} G_{11} + H_{21} G_{21}) \right]
+ \Omega_1 \Omega_2 (H_{22} G_{12} + H_{12} G_{22})^2
- \Omega_1 \Omega_2 \left[ H_{22} (G_{11} - G_{21}) - G_{12} (H_{11} - H_{21}) \right]
\times \left[ H_{12} (G_{11} - G_{21}) - G_{12} (H_{11} - H_{21}) \right]
/ 64 (H_{11} H_{21} - H_{22} H_{12}) (H_{11} - H_{21})^2 + 4 H_{22} H_{12}^2.
\]

(87)

Similar results can also be obtained for the case with \( \omega \) near \((\Omega_1 - \Omega_2)/2\) by changing the sign of \( \Omega_2 \).

6.3. Numerical illustration for the general deformations

We present a numerical discussion of the stability behavior where the surface deflections \( \xi (1) \) and \( \xi (2) \) are independent. Two versions of the stability behavior are taken into account: the stability behavior for static electric fields and the stability discussion in the resonant cases, where the field has an oscillating manner with frequency \( \omega \).

In the calculations the computed value of the surface charge density parameter \( \sigma_2^2 \) versus the wavenumber \( k \) is utilized. In these calculations all the physical parameters are sought in the dimensionless form as defined before. Fixing the value of all the physical parameters (the field frequency, the sheet thickness, the velocity ratios, the viscosity ratios, and the relaxation times ratios) will make the stability examination except one parameter having consequence value for comparison. The parameters are changed one by one. In the numerical calculations use of the same system as in Fig. 1 is made, where \( E_{00}^{(1)} = 30, E_{00}^{(2)} = 8.3, \) and \( E_{00}^{(3)} = 20. \) Despite this system, the numerical calculations were made for the stability conditions (74), (75), and (76), which control the case of the static case, while in the resonant case the calculations are made for the stability condition (82). Condition (82) refers to the resonant case of \( \omega \approx \Omega_1 \). The computation showed that Eq. (85) has two complex roots. Consequently, the two resonances \( \omega \approx 1/2 (\Omega_1 + \Omega_2) \) and \( \omega \approx 1/2 (\Omega_1 - \Omega_2) \) have no effect on the stability of the system.

The influence of the sheet thickness \( a \), the velocity ratio \( \mu_0 \), the viscosity ratio \( \mu_0 \), and the elasticity ratio \( \lambda_0 \) on the stability picture where the field is taken in the static case are displayed in Figs. 10–13, respectively. The corresponding influences in the resonant case of \( \omega \approx \Omega_1 \) are displayed in Figs. 14–18.

In Figs. 10–13 we plotted \( \log(\sigma_2^2) \) versus the wavenumber \( k \) for the transition curves (77)–(79). Due to the numerical parameter used here, the calculation shows that \( \sigma_2^2 \) has negative values as well as the curve \( \sigma_2^2 \). The positive values of \( \sigma_1^2 \), \( \sigma_2^2 \), and \( \sigma_3^2 \) are displayed in the graphs. Accordingly, the stability occurs when

\[
\sigma_j^2 > \sigma_{1j}^2 \quad \text{and} \quad \sigma_j^2 < \sigma_{1j}^2
\]

or

\[
\sigma_j^2 < \sigma_{2j}^2 \quad \text{and} \quad \sigma_j^2 < \sigma_{4j}^2, \quad \sigma_{1j}^2 > \sigma_{2j}^2.
\]

Three consequent numerical values for each parameter mentioned above are given. The low numerical value is represented by the symbol \( \circ \) in the graphs, the mid numerical value is denoted by the symbol \( \ast \), and the high numerical value is indicated by the symbol \( \times \).

In Fig. 10 we examine the influence of the sheet thickness \( a \) on the stability behavior. For a specific value of the sheet thickness, the stability appears except at a gap bounded by the two transition curves \( \sigma_{1j}^2 \) and \( \sigma_{2j}^2 \), while the instability appears for large values of \( \sigma_2^2 \). The variation of the parameter \( a \) from \( a = 0.1 \) to \( a = 0.3 \) and then to \( a = 0.5 \) affected

Fig. 10. Stability diagram for the influence of the fluid sheet thickness on the stability criteria, for the same system as in Fig. 1. The curves of the symbol \( \circ \) refer to the case of \( a = 0.1 \), the curves of the symbol \( \ast \) represent the case of \( a = 0.3 \) while the curves of the symbol \( \times \) denote the case of \( a = 0.5 \). The graph indicates the transition curves (84)–(86).

Fig. 11. Effect of the velocity ratio \( \mu_0 \) on the stability picture for the same system as in Fig. 1. The curves labeled by the symbol \( \circ \) refer to the case of \( \mu_0 = 0.5 \), the curves marked by the symbol \( + \) refer to the case of \( \mu_0 = 1 \), while the curves of the symbol \( \times \) denote the case of \( \mu_0 = 1.5 \).
same system as in Fig. 10 except that $E_0^{(1)} = 3$, $E_0^{(2)} = 0.83$, $E_0^{(3)} = 2$, $Q_1 = 50$, and $Q_2 = 35$. In these calculations two resonant points are found. The transition curves that are embedded from these resonance points have bounded the unstable resonance regions. The two unstable regions are connected and collected into one unstable region. It is observed that the first resonance point lies at $k = 0.2156$ and this point has not been affected by the variation of the parameters $\omega$, $\nu_0$, $\nu_0$, while a very small shift occurs due to the variation of $\mu_0$ and $\lambda_0$. The second resonance point is found as a function of both $\omega$, $\nu_0$, $\mu_0$, and $\lambda_0$.

It is noted that these points are evaluated from the equation $\omega = \Omega_1$; i.e.,

$$\omega = \frac{1}{2} \left\{ (H_{11} + H_{22}) + \sqrt{(H_{11} - H_{22})^2 + 4H_{12}H_{22}} \right\}.$$  

There are two stable regions. The first stable region, $S_1$, lies between the two resonant points and is bounded by the transition curve $\sigma^{**}/3$. The second stable region, $S_2$, lies outside the two resonance points and outside the transition curve $\sigma^{**}$.

In Figs. 14–18 the symbol o refers to the low numerical value, the symbol * refers to the mid numerical value, and the symbol x refers to the high numerical value for each of the values of the parameters $\omega$, $\alpha$, $\nu_0$, $\mu_0$, and $\lambda_0$.

The influence of the field frequency $\omega$ on the stability at the resonance case is displayed in Fig. 14. In this graph the increase in the field frequency leads to the second resonance point to the direction of increasing the $k$-axis, associated with moving both the transition curves $\sigma^{**}$ and $\sigma^{**}/3$ up. Consequently, the stable region $S_1$ increases in its width. This shows the stabilizing influence of increasing $\omega$. Because the movement of the curve $\sigma^{**}$ is greater than the movement of the curve $\sigma^{**}/3$ the unstable region $U$ has increased. This means that the increase of $\omega$ has a destabilizing influence. From this discussion we observe that the increase of $\omega$ plays a dual role in the stability behavior in the resonant case. The major influence of increasing $\omega$ is a destabilizing influence.

The effect of small variation of the sheet thickness is illustrated in Fig. 15. It is shown that a small increase in the parameter $a$ has affected the stability picture in the resonance case of $\omega = \Omega_1$. As $a$ is increased, the second resonance point has shifted in the direction of decreasing $k$-axis. In addition, the two transition curves have moved down. Consequently, both the unstable region $U$ and the stable region $S_1$ have decreased in width. Thus two roles are found, a stabilizing influence in the $U$-region and a destabilizing influence in the $S_1$-region. Moreover, the $S_2$-region has increased. The major influence of increasing $a$ is the stabilizing influence. The destabilizing influence was observed before in case of anti-symmetric examination.

The influence of a slight increase in the velocity ratio $v_0$ is illustrated in Fig. 16. The computation is as in Fig. 10 except that $v_0$ has three different numerical values, $v_0 = 1.2, 1.22$, and 1.24. Inspection of the graph shows that the slight in-
\[ \begin{align*}
\times (e^{i(k_1-k_2)y} + e^{i(k_1+k_2)y}) \\
- \frac{1}{4 \sinh k \alpha} \left[ -\frac{e^{i\gamma_1}}{e^{i\gamma_2}} \cos \omega t (\gamma_1 - \gamma_2) + \frac{1}{Q \sigma \phi \cos \omega t} \right] \\
\times \left[ 2 \left( 1 - \lambda^{(2)} \left( \frac{\partial}{\partial t} + v_0^{(2)} ik \right) \right) \mu^{(2)} (\gamma_1 + v_0^{(2)} ik \gamma_1) \\
- 2 \left( 1 - \lambda^{(1)} \left( \frac{\partial}{\partial t} + v_0^{(1)} ik \right) \right) \mu^{(1)} (\gamma_1 + v_0^{(1)} ik \gamma_1) \right] \\
\times \left( e^{i(k_1-k_2)y} - e^{i(k_1+k_2)y}, \ |y| < a, \right)
\end{align*} \]

The coefficients in Eq. (39) are

\[ a_{1j} = \frac{2(-1)^{j+1} k^2}{\phi^{(1)}(0)^{(2)} - \phi^{(2)}(0)^{(1)} E_0^{(2)} - \phi^{(2)}(0)^{(2)} - \phi^{(3)}(0)^{(2)} E_0^{(3)}} \\
\times \left( e^{(2)(j-1)} E_0^{(2)(j-1)} - \phi^{(2)}(0)^{(j)} E_0^{(2)} - \phi^{(3)(j)} E_0^{(3)} \right) \\
\times \left( e^{(2)(j-1)} E_0^{(2)(j-1)} - \phi^{(2)}(0)^{(j)} E_0^{(2)} - \phi^{(3)(j)} E_0^{(3)} \right) \\
\times \left( e^{(2)(j-1)} E_0^{(2)(j-1)} - \phi^{(2)}(0)^{(j)} E_0^{(2)} - \phi^{(3)(j)} E_0^{(3)} \right) \]

\[ a_{2j} = \frac{2(-1)^{j+1} k^2}{\phi^{(1)}(0)^{(2)} - \phi^{(2)}(0)^{(1)} E_0^{(2)} - \phi^{(2)}(0)^{(2)} - \phi^{(3)}(0)^{(2)} E_0^{(3)}} \\
\times \left( e^{(2)(j-1)} E_0^{(2)(j-1)} - \phi^{(2)}(0)^{(j)} E_0^{(2)} - \phi^{(3)(j)} E_0^{(3)} \right) \\
\times \left( e^{(2)(j-1)} E_0^{(2)(j-1)} - \phi^{(2)}(0)^{(j)} E_0^{(2)} - \phi^{(3)(j)} E_0^{(3)} \right) \]

\[ b_{1j} = \frac{2k \phi^{(2)}(0)^{(2)} E_0^{(2)} - \phi^{(3)}(0)^{(2)} E_0^{(3)}}{8k^4} \\
\times \left( e^{(2)(j-1)} E_0^{(2)(j-1)} - \phi^{(2)}(0)^{(j)} E_0^{(2)} - \phi^{(3)(j)} E_0^{(3)} \right) \\
\times \left( e^{(2)(j-1)} E_0^{(2)(j-1)} - \phi^{(2)}(0)^{(j)} E_0^{(2)} - \phi^{(3)(j)} E_0^{(3)} \right) \]

\[ b_{2j} = \frac{2k \phi^{(2)}(0)^{(2)} E_0^{(2)} - \phi^{(3)}(0)^{(2)} E_0^{(3)}}{8k^4} \\
\times \left( e^{(2)(j-1)} E_0^{(2)(j-1)} - \phi^{(2)}(0)^{(j)} E_0^{(2)} - \phi^{(3)(j)} E_0^{(3)} \right) \\
\times \left( e^{(2)(j-1)} E_0^{(2)(j-1)} - \phi^{(2)}(0)^{(j)} E_0^{(2)} - \phi^{(3)(j)} E_0^{(3)} \right) \]
\[ \ Transcript the mathematical equations here. \]

The formula for the coefficients appeared in Eq. (65) are

\[
\tilde{H}_{j1} = e_{j1} + \frac{1}{a_{11}d_{21} - a_{22}d_{21}} [b_{j1}(a_{12}d_{11} - a_{22}d_{21}) + b_{j2}(a_{11}d_{21} - a_{21}d_{11})],
\]

\[
\tilde{H}_{j2} = c_{j2} + \frac{1}{a_{11}d_{22} - a_{22}d_{21}} [b_{j1}(a_{12}d_{12} - a_{22}d_{12}) + b_{j2}(a_{11}d_{22} - a_{21}d_{22})].
\]

References